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Math 412

Homework #2

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Problem: Show that $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, is ordered but not well-ordered.

Answer: A set with a binary relation R on its elements that is *reflexive*, *antisymmetric*, and *transitive* is described as a partially ordered set. The relation is a *total order* if for all element a, b either aRb or bRa holds.

Two polynomials are considered to be *equal* $(=)$ if and only if the corresponding coefficients for each power of x are equal. We also define a polynomial $p(x)$ is considered *less than* $(<)$ a polynomial $q(x)$ if and only if $a_k < b_k$, where a_k and b_k are the coefficients in $p(x)$ and $q(x)$ respectively, and k is the largest such that the x^k terms in $p(x)$ and $q(x)$ are different. For example, we have the following relation.

$$p_1(x) = 8 + 2x + x^2$$

$$p_2(x) = 5 + 3x + x^3$$

$$p_3(x) = 1 + 4x + x^2$$

$$\Rightarrow p_2(x) > p_3(x) > p_1(x)\ .$$

Now we can show the relation $(\mathbb{Z}[x], \leq)$ satisfies the three poset requirements. The relation is reflexive by the equality definition. The relation is antisymmetric according to the less than relationship described above. The relation is obvious transitive. $p_1(x) < p_2(x)$ and $p_2(x) < p_3(x)$ derive $p_1(x) < p_3(x)$. The relation is also a total ordering. Therefore, the ring of polynomials with integer coefficients is *ordered*.

A well-order relation (or well-ordering) on a set S is a total order on S with the property that every non-empty subset of S has a least element in this ordering. The ring of polynomials with integer coefficients is *not well-ordered* because the standard ordering $\leq$ of the integers, $\mathbb{Z}$, is not a well ordering. For example, the set of negative integers does not contain a least element hence integers, $\mathbb{Z}$, is not well-ordered. Since the smallest $\mathbb{Z}[x]$'s would consist only one integer coefficient, $p_{small} = z$, where $z \in \mathbb{Z}$, and there does not exist the smallest integer z.

Extra: Show that $\mathbb{Z}[x]$, the polynomials with integer coefficients, forms a ring.

Answer: First, we show that $\mathbb{Z}[x]$ is an abelian group under polynomial addition. The zero polynomial, $p(x) = 0$, is the additive identity. For a given polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$, the additive inverse of $p(x)$ is simply,

$$\begin{aligned} -p(x) &= \sum_{i=0}^{n} (-a_i) x^i \\ &= -\sum_{i=0}^{n} a_i x^i \,. \end{aligned}$$

Commutativity is obvious,

$$\begin{aligned} p(x) + q(x) &= \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i \\ &= \sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{n} a_i x^i \\ &= q(x) + p(x) \,. \end{aligned}$$

Associativity is also clear.

$$\begin{aligned} [p(x) + q(x)] + r(x) &= \left(\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i\right) + \sum_{i=0}^{n} c_i x^i \\ &= \sum_{i=0}^{n} (a_i + b_i) x^i + \sum_{i=0}^{n} c_i x^i \\ &= \sum_{i=0}^{n} (a_i + b_i + c_i) x^i \\ &= p(x) + [q(x) + r(x)] \,. \end{aligned}$$

Second, we show that the associativity in respect to the polynomial multiplication.

$$\begin{aligned} [p(x) \cdot q(x)] \cdot r(x) &= \left(\sum_{i=0}^{m} a_i x^i \cdot \sum_{i=0}^{n} b_i x^i\right) \cdot \sum_{i=0}^{p} c_i x^i \\ &= \left[\sum_{i=0}^{m+n} \left(\sum_{j+k=i} a_j b_k\right) x^i\right] \cdot \sum_{i=0}^{p} c_i x^i \\ &= \sum_{i=0}^{m+n+p} \left(\sum_{j+k+l=i} a_j b_k c_r\right) x^i \\ &= p(x) \cdot [q(x) \cdot r(x)] \,. \end{aligned}$$

Finally, we show that the polynomial multiplication operation is distributive over the polynomial addition operation.

$$
\begin{aligned}
p(x)\cdot[q(x)+r(x)] &= \sum_{i=0}^{m} a_i x^i \cdot \left(\sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{n} c_i x^i\right) \\
&= \sum_{i=0}^{m} a_i x^i \cdot \sum_{i=0}^{n} (b_i + c_i) x^i \\
&= \sum_{i=0}^{m+n} \left(\sum_{j+k=i} a_j (b_k + c_k)\right) x^i \\
&= \sum_{i=0}^{m+n} \left(\sum_{j+k=i} a_j b_k + a_j c_k\right) x^i \\
p(x)\cdot q(x) + p(x)\cdot r(x) &= \sum_{i=0}^{m} a_i x^i \cdot \sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{m} a_i x^i \cdot \sum_{i=0}^{n} c_i x^i \\
&= \sum_{i=0}^{m+n} \left(\sum_{j+k=i} a_j b_k\right) x^i + \sum_{i=0}^{m+n} \left(\sum_{j+k=i} a_j c_k\right) x^i \\
&= \sum_{i=0}^{m+n} \left(\sum_{j+k=i} a_j b_k + a_j c_k\right) x^i
\end{aligned}
$$

$$
\Rightarrow p(x)\cdot[q(x)+r(x)] = p(x)\cdot q(x) + p(x)\cdot r(x)\,.
$$

In summary, we've shown that $\mathbb{Z}[x]$ forms an abelian group, it satisfies the multiplication associativity law, and it multiplication operation is distributive over addition. Therefore, $\mathbb{Z}[x]$ forms a ring.