Section 1.8
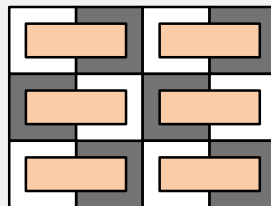
Exercise 1: Show that an m-by-n chessboard has a perfect cover by dominoes if and only if at least one of m and n is even.

Answer: *Step 1*. Suppose one of m and n is even and the other one is odd. Since we can cover rows with even number of blocks, we can cover the entire chessboard. For example, a 4-by-3 chessboard:

Step 2. Suppose both m and n are even. Again since we can cover rows with even number of blocks, we can cover the entire chessboard. For example, a 4-by-4 chessboard:
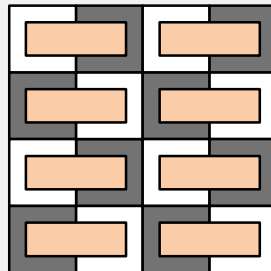
Step 3. Assume both m and n are odd, and the chessboard has a perfect cover. Since the numbers of two color blocks are different, and one domino covers exactly one of the each, the chessboard can't have a perfect cover. This is a conflict. Therefore, the assumption is wrong. When both m and n are odd, the chess board doesn't have a perfect cover. In other words, if the chessboard has a perfect cover, at least one of m and n is even. For example, a 3-by-3 chessboard:
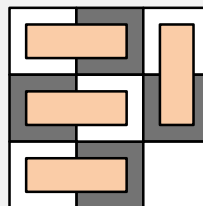
Combining *Step 1*, *Step 2*, and *Step 3*, we've shown both side of "if and only if". Therefore an m-by-n chessboard has a perfect cover by dominoes is the necessary and sufficient condition for at least one of m and n is even, and vice versa.

Exercise 2: Consider an m-by-n chessboard with m and n both odd. To fix the notation, suppose that the square in the upper left-hand corner is colored white. Show that if a white square is cut out anywhere on the board, the resulting pruned board has a perfect cover by dominoes.
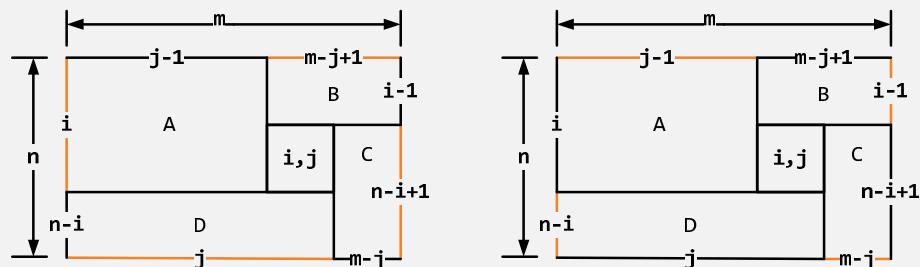
Answer: If we index the blocks as the two dimensional array, the sum of indexes for white blocks is an even number, giving both m and n are odd. For example, a 5-by3 chessboard:

Suppose we have white block $W_{i,j}$ cut out. We can picture the chessboard as below. It consists four rectangular boards and a missing white block.
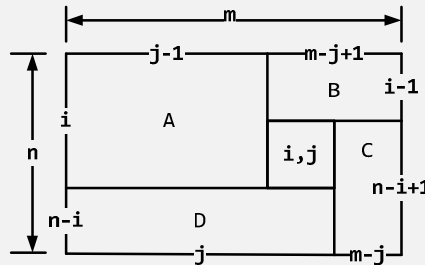
As we discussed in the first paragraph, the indexes i, j are either both even or both odd. If they are both even, then in rectangular A, we have one edge (i) being an even number; in rectangular B, we have one edge $(m - j + 1)$ being an even number; in rectangular C, we have one edge $(n - i + 1)$ being an even number; in rectangular D, we have one edge (j) being an even number. If the indexes i, j are both odd, then the other set of edges in each rectangular are even. They are shown as below. They orange edges are even.


Both i, j are even   Btohi, j are odd

We've proved in Exercise 1 that an m-by-n chessboard has a perfect cover by dominoes if and only if at least one of m and n is even. Therefore all generated rectangular (A, B, C, D) have prefect covers. In other words, an m-by-n chessboard with m and n both odd and with one extra white block anywhere on the board cut off would end with a board that can has perfect cover by dominoes.

Exercise 3:   Imagine a prison consisting of 64 cells arranged like the squares of an 8-by-8 chessboard. There are doors between all adjoining cells. A prisoner in one of the corner cells is told that he will be released, provided he can get into the diagonally opposite corner cell after passing through every other cell exactly once. Can the prisoner obtain his freedom?

Answer: No, the prisoner can't obtain his freedom. We can consider the prison cells together as a chessboard map. The path this prisoner takes consists of alternating white and black blocks. Assume the prisoner start with a white block. In order to exit from the opposite corner cell, the path needs to end with a white block as well. The path can be shown as below:
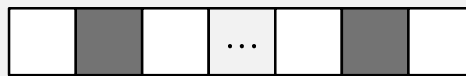

In this case, the path would contain one more white block than black block, $n_{white} = n_{black} + 1$. The 8-by-8 chessboard, however, contains the same number of white and black blocks, $n_{white} = n_{black} = 32$. Therefore a path like the one shown above can't happen, given that it covers each block once and only once.

Exercise 4a: Let $f(n)$ count the number of different perfect covers of a 2-by-n chessboard by dominoes. Evaluate $f(1), f(2), f(3), f(4)$, and $f(5)$. Try to find (and verify) a simple relation that the counting function f satisfies. Use this relation to compute $f(12)$.
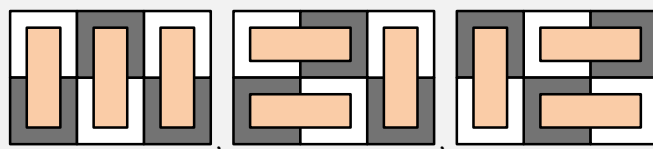
Answer: For base cases, we can count the number of perfect covers of a 2-by-n chessboard by dominoes.
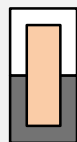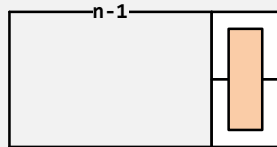

$$f(1) = 1$$
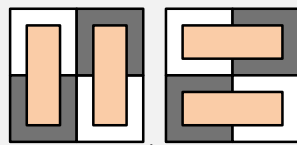
$$f(2) = 2$$

$$f(3) = 3$$

We propose the recursive relationship of function f is $f(n) = f(n-1) + f(n-2)$, when $n > 2$. The base cases were shown as above $f(3) = f(1) + f(2)$. The proof for the general case is given below.

If the added cells go together and the layout of the previous 2-by-$n-1$ map with dominoes stays the same, then there is one way to place a domino to cover the added column.

$f(n-1)$ ways

If the added cells go together with their left side neighbors instead with each other and the layout of the previous 2-by$n-2$ map with dominoes stays the same, then there is one way to place a domino to cover the added column.
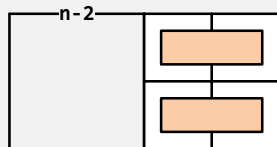

$f(n-2)$ ways

Since the map is given as 2-by-n, these are the only two possible ways to cover the added column. Therefore we've shown $f(n) = f(n-1) + f(n-2)$. Having this conclusion, we can compute the values by applying this recursive relation.

$$f(4) = f(3) + f(2) = 3 + 2 = 5$$
$$f(5) = f(4) + f(3) = 5 + 3 = 8$$
$$f(6) = f(5) + f(4) = 8 + 5 = 13$$
$$f(7) = f(6) + f(5) = 13 + 8 = 21$$
$$f(8) = f(7) + f(6) = 21 + 13 = 34$$
$$f(9) = f(8) + f(7) = 34 + 21 = 55$$
$$f(10) = f(9) + f(8) = 55 + 34 = 89$$
$$f(11) = f(10) + f(9) = 89 + 55 = 144$$
$$f(12) = f(11) + f(10) = 144 + 89 = 233.$$

**Exercise 5:** Find the number of different perfect covers of a 3-by-4 chessboard by dominoes.

Answer: If we consider the domino layout related to the last row is a varying condition. The following 5 layouts are the possibilities that the bottom row can be covered by dominos in the 3-by-4 chessboard map. Then we can evaluate one at a time. We will use $f(m, n)$ to denote the number of perfect covers of an m-by-n chessboard map.
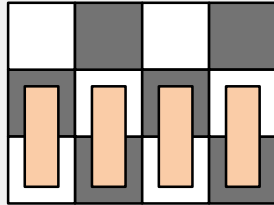
Only one way
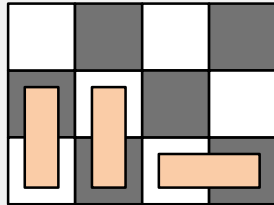
$f(2,2) = 2$ ways
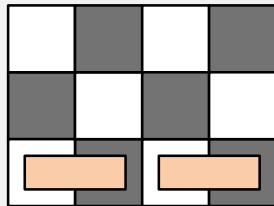
$f(2,4) = 5$ ways
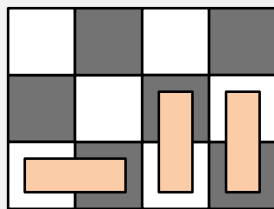
$f(2,2) = 2$ ways
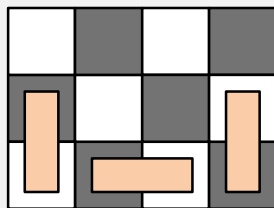
Only one way

Sum them up, there are $1 + 2 + 5 + 2 + 1 = 11$ prefect covers of a 3-by-4 chessboard by dominoes.

Section 3.4

Exercise 4:   Show that if $n + 1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there are always two which differ by 1.

Answer: We can divide the given set of integers into n sets, and they are $\{1, 2\}, \{3, 4\}, \cdots, \{2n - 1, 2n\}$. Now we pick $n + 1$ integers from these n sets. According to Pigeonhole Theorem if $n + 1$ objects

are distributed into n boxes, then at least one box contains two or more of the objects, at least two integers would be chosen from a same set. Then these two integers would be different by 1.

Exercise 5: Show that if $n + 1$ integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always two which differ by at most 2.

Answer: We can divide the given set of integers into n sets, and they are $\{1, 2, 3\}, \{4, 5, 6\}, \cdots, \{2n - 2, 2n - 1, 2n\}$. Now we pick $n + 1$ integers from these n sets. According to Pigeonhole Theorem if $n + 1$ objects are distributed into n boxes, then at least one box contains two or more of the objects, at least two integers would be chosen from a same set. Then these two integers would be different by either 1 or 2, so at most 2.

Exercise 9: In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old.

    a. Prove that we can always find two groups of people (with no common person) the sum of whose ages is the same.

Answer: If there are people with the same age, then we are done. They can each be a group, and we've found groups of people with the same age sum. Now we continue with the condition that nobody has the same age as anyone else in these 10 people.

Among 10 people, there are $2^{10} - 1$ that many ways of forming groups. This number is derived by the following logic. For each person, he's either selected or not in a particular group, so there are 2^{10} different selections. While it doesn't count if the group contains nobody, so the number of groups is $2^{10} - 1 = 1023$.

Since they are between 1 and 60 years old, and all different, the maximum sum of group age is $60 + 59 + 58 + 57 + 56 + 55 + 54 + 53 + 52 + 51 = 111 \times 5 = 555$. The minimum sum of group age is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 11 \times 5 = 55$. In between of the maximum sum and the minimum sum, there are $555 - 55 + 1 = 501$ that many possible age sums.

Therefore there are 1023 ways of choosing groups, while there are only 501 possible age sums. According the Pigeonhole Theorem, we can always find two groups of people that have the same age sum.

    b. Can 10 be replaced by a smaller number?

Exercise 27: A collection of subsets of $\{1, 2, \cdots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

Answer: If there is no restrictions, the collection of subset of $\{1, 2, \cdots, n\}$ has at most 2^n elements because for each integer in $\{1, 2, \cdots, n\}$, we either select it or not. This includes the empty subset.

Within these 2^n subsets, if we randomly pick a set A, there is another set A^- containing exactly the opposite integers, since no restriction was enforced when we make the subsets. We put A and its complementary set A^- together. We repeat this procedure until all 2^n subsets are grouped together two by two. Obviously there are 2^{n-1} groups, and each group contains 2 subsets that do not contain any integers in common with each other.

Now if we select $2^{n-1} + 1$ subsets from these 2^{n-1} pairs, according to the Pigeonhole Theorem, at least two of them would come from the same pair. These two subsets would, therefore, be complementary with each other and not sharing any integers. So, the assumption is not right. We can't have anything more than 2^{n-1} subsets in order to satisfy the condition that each pair of subsets has at least one integer in common.

Exercises on Ramsey Theory

Question 2: Show that $R(3; 4) > 8$. This says that there is some way of coloring every edge of an 8 point set with either red or blue such that there is no red triangle nor is there a set of four points with all connecting edges blue. Here is the graph that works: take 8 points and arrange them around a circle. For each point color the lines to both its left and right neighbors red and also color the line to the dead opposite point red. All other lines are blue. Show this works. (Orange indicates red, and black indicates blue in the following graph.)
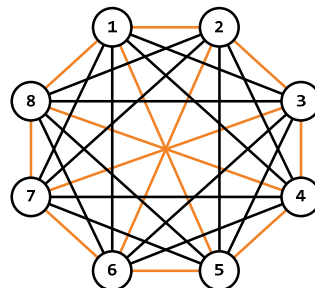
Answer:

In order to show the graph above has neither a red triangle nor a blue quadruple (6 lines), we can show that we would fail in the procedure of trying to look for a red triangle or a blue quadruple.

Looking for a red triangle: Since every vertex is the same as any of the other vertices. We start with one particular vertex (#1) and look for a red triangle containing vertex #1. If there's no red triangle with vertex #1, there is no red triangle with any other vertex. In other word, no red triangle exists in the given graph.

Vertices #3, #4, #6, and #7 are connected with vertex #1 with blue edges, so they are excluded. We have vertices #2, #5, and #8 left. But any two of these three are connected with a blue edge. Therefore, we've shown there's no red triangle in the given graph.

Looking for a blue quadruple: Since every vertex is the same as any of the other vertices. We start with one particular vertex (#1) and look for a blue quadruple containing vertex #1. If there's no blue quadruple with vertex #1, there is no blue quadruple with any other vertex. In other word, no blue quadruple exists in the given graph.

Vertices #2, #5, and #8 are connected with vertex #1 with red edges, so they are excluded. We have vertices #3, #4, #6, and #7 left. We group vertices #3 and #4 together as a set, vertices #6 and #7 together as a set. In each set, the items are connected with a red edge. We need to select 3 vertices out of 4 to form a quadruple with vertex #1. According the Pigeonhole Theory, selecting 3 items out of 2 sets, 2 of the selections would be from the same set. So, two of the selected vertices would be connected by a red edge. In other words, we've shown there's no blue quadruple in the given graph.

Conclude from the two proofs, we can neither find any red triangles nor any blue quadruples in the graph given. Therefore, we've shown $R(3, 4) > 8$.

Question 3:

I will show in class that $R(3; 4) = 9$. Using this and the fact shown in class:

$$R(m, n) \leq R(m - 1, n) + R(m, n - 1)$$

Prove that $R(3, 5) \leq 14$.

Answer:

According to the given fact above, and the fact that $R(2, m) = R(m, 2) = m$, we have the following relation.
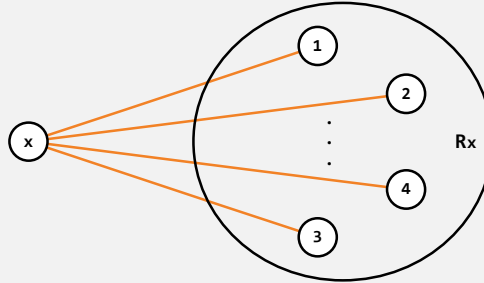
$$R(3, 5) \leq R(2, 5) + R(3, 4)$$

$$\leq 5 + R(3, 4)$$

$$\leq 5 + 9$$

$$\leq 14 \,.$$

Question 4:   Prove Lemma 2. If $|R_x| \geq 4$, then the graph either has a red triangle or a blue quadruple.

Answer: $|R_x|$ indicates the number of vertices that are connected to vertex x with a red line. Since this number is at least four, we can represent this relation using the following illustration.


We know that $R(2,4) = 4$. That is saying, there either exists a red line or a blue quadruple in a complete graph with four vertices. Consider the vertices #1, #2, #3, and #4 make up a complete graph.

In case 1: There is a red line. Then we have a red triangle after connecting any of the two vertices in R_x, since both of them would be connected to vertex x with red edges.
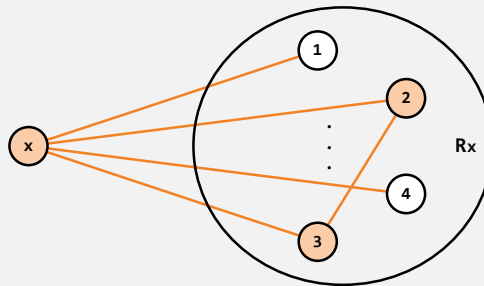
In case 2: There is a blue quadruple. Then we're done.
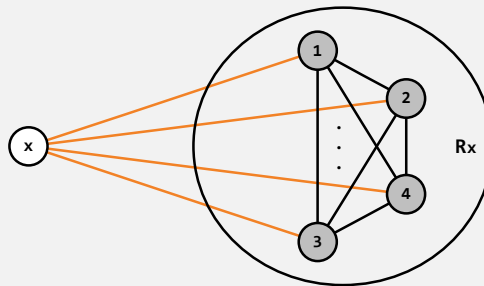
Conclude from the two cases above, we've shown that if $|R_x| \geq 4$, then the graph either has a red triangle or a blue quadruple – Lemma 2.