Student: Yu Cheng (Jade)

Math 475

Homework #2

February 14, 2010

Section 2.7

Exercise 2: How many orderings are there for a deck of 52 cards if all the cards of the same suit are together?

Answer:

Number of ways to deal one suit cards:	13!
Number of different suits:	4
Number of ways to order different suits	4!
Total number of ordering	13! 13! 13! 13! 4! .

Exercise 5: Determine the largest power of 10 that is a factor of the following numbers

a. 50!

Answer: One factor 2 and one factor 5 multiply together give one factor of 10, which is $50! = 10^n a_1^{b_1} a_2^{b_2} \cdots a_k^{b_k} = 2^{n_1} 5^{n_2} a'^{b'_1}_1 a'^{b'_2}_2 \cdots a'^{b'_k}_k$. Since factor 2 out numbers factor 5, we only need to count the number of factor 5's.

$5^1 \times 10 \leq 50$	10 numbers within 50 contribute at least one 5
$5^2 \times 2 \leq 50$	2 numbers within 50 contribute two 5's

Therefore there are 2 numbers contributing two factor 5's and $10 - 2 = 8$ numbers contributing one factor 5. Therefore there are $2 \times 2 + 8 \times 1 = 12$ factor 5's. Hence, there are 12 zeros in 50!.

b. 1000!

Answer: One factor 2 and one factor 5 multiply together give one factor of 10, which is $1000! = 10^n a_1^{b_1} a_2^{b_2} \cdots a_k^{b_k} = 2^{n_1} 5^{n_2} a'^{b'_1}_1 a'^{b'_2}_2 \cdots a'^{b'_k}_k$. Since factor 2 out numbers factor 5, we only need to count the number of factor 5's.

$5^1 \times 200 \leq 1000$	200 numbers within 1000 contribute at least one 5
$5^2 \times 40 \leq 1000$	40 numbers within 1000 contribute at least two 5's
$5^3 \times 8 \leq 1000$	8 numbers within 1000 contribute at least three 5's
$5^4 \times 1 \leq 1000$	1 numbers within 1000 contributes four 5's

Therefore there are 1 number contributes four 5's, $8-1=7$ numbers contribute three 5's, $40-1-7=32$ numbers contribute two 5's, and $200-1-7-32=160$ numbers contribute one 5 factor. Therefore there are $1\times 4+7\times 3+32\times 2+160\times 1=249$ factor 5's. Hence, there are 249 zeros in 1000!.

Exercise 10: A committee of five people is to be chosen from a club that boasts a membership of 10 men and 12 women.

a. How many ways the committee can be formed if it is to contain at least two women?

Answer:

Ways to form the committee if there are no constraints	$\binom{22}{5}$
Ways to form the committee if no women were chosen	$\binom{10}{5}$
Ways to form the committee if one woman was chosen	$\binom{10}{4}\binom{12}{1}$
Therefore the answer is	$\binom{22}{5}-\binom{10}{5}-\binom{10}{4}\binom{12}{1}=23562$.

b. How many ways if, in addition, one particular man and one particular woman who are members of the club refuse to serve together on the committee?

Answer:

Ways to form the committee without the new constraint $\binom{22}{5}-\binom{10}{5}-\binom{10}{4}\binom{12}{1}=23562$

Ways to form the committee selecting the particular man and women, and selecting at least two women:

One other woman (Two other men)	$\binom{9}{2}\binom{11}{1}$
Two other women (One other man)	$\binom{9}{1}\binom{11}{2}$
Three other women (No other men)	$\binom{9}{0}\binom{11}{3}$

Therefore the answer is $23562-\binom{9}{2}\binom{11}{1}-\binom{9}{1}\binom{11}{2}-\binom{9}{0}\binom{11}{3}=22506$.

Exercise 14: A classroom has two rows of eight seats each. There are 14 students, 5 of who always sit in the front row and 4 of who always sit in the back row. In how many ways can the students be seated?

Answer:

Ways to seat the 5 who always sit in the front row	$P(8,5)$
Ways to seat the 4 who always sit in the back row	$P(8,4)$
Ways to seat the rest of the students	$P(16-9,\ 14-9)=P(7,5)$
Therefore the answer is	$P(8,5)P(8,4)P(7,5)=\ 28449792000$.

Exercise 21-a: How many permutations are there of the letters of the word ADDRESSES?

Answer:

Permutations of 9 letters	$P(9,9)$
Permutations of the two 'D' letters	$P(2,2)$

Permutations of the two 'E' letters	$P(2,2)$
Permutations of the three 'S' letters	$P(3,3)$
Therefore the answer is	$\frac{P(9,9)}{P(2,2)P(2,2)P(3,3)} = \frac{9!}{2!2!3!} = 15120$.

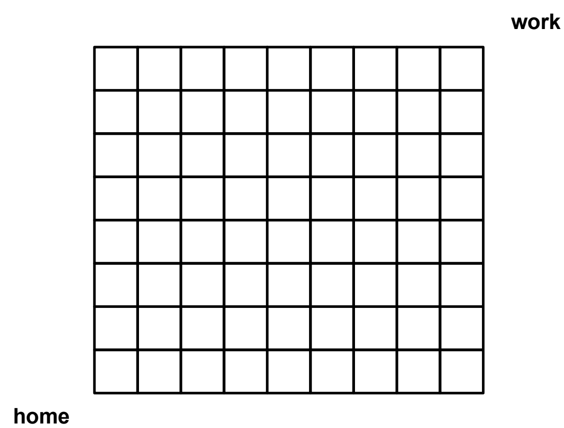
b. How many 8-permutations are there of these nine letters?

Answer:

8-permutations without a letter 'A'	$\frac{P(8,8)}{P(2,2)P(2,2)P(3,3)}$
8-permutations without a letter 'D'	$\frac{P(8,8)}{P(2,2)P(3,3)}$
8-permutations without a letter 'R'	$\frac{P(8,8)}{P(2,2)P(2,2)P(3,3)}$
8-permutations without a letter 'E'	$\frac{P(8,8)}{P(2,2)P(3,3)}$
8-permutations without a letter 'S'	$\frac{P(8,8)}{P(2,2)P(2,2)P(2,2)}$

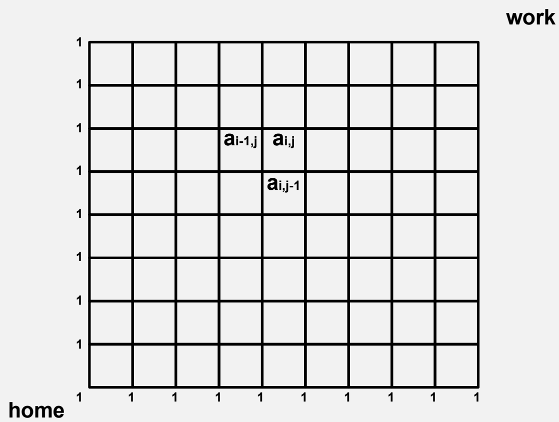
Therefore the answer is $2 \times \frac{P(8,8)}{P(2,2)P(2,2)P(3,3)} + 2 \times \frac{P(8,8)}{P(2,2)P(3,3)} + \frac{P(8,8)}{P(2,2)P(2,2)P(2,2)} = 15120$.

Exercise 28: A secretary works in a building located nine blocks east and eight blocks north of his home. Every day he walks 17 blocks to work.



a. How many different routes are possible for him?

Answer: Since it takes this person only 17 blocks to get from home to work, it means he can go only eastern wards and northern wards on the way. Therefore the ways to get to one intersection is determined by the ways to get to the previous two intersections where this person can come from. Specifically, we can initialize the graph as shown in the following graph, $a_{i,0} = 1$, $a_{0,j} = 1$. And compute the numbers of different ways to get to intersections as $a_{i,j} = a_{i-1,j} + a_{i,j-1}$.



Now we are able to compute the number of ways to get to any intersection. And we also know the goal is to compute $a_{9,8}$. Note that the formula $a_{i,j} = a_{i-1,j} + a_{i,j-1}$ falls in the pattern of Pascal Triangle. The mapping looks like:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array} \quad \Rightarrow \quad \begin{array}{ccccccccc} &&&&a_{0,0}&&&& \\ &&&a_{1,0}&&a_{0,1}&&& \\ &&a_{2,0}&&a_{1,1}&&a_{0,2}&& \\ &a_{3,0}&&a_{2,1}&&a_{1,2}&&a_{0,3}& \\ a_{4,0}&&a_{3,1}&&a_{2,2}&&a_{1,3}&&a_{0,4} \end{array}$$

We also know that the Pascal Triangle map to the combination expressions:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array} \quad \Rightarrow \quad \begin{array}{ccccccccc} &&&&\binom{0}{0}&&&& \\ &&&\binom{1}{0}&&\binom{1}{1}&&& \\ &&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&& \\ &\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}& \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \end{array}$$

We are now looking for the solutions for $a_{9,8}$, we can write the corresponding row as:

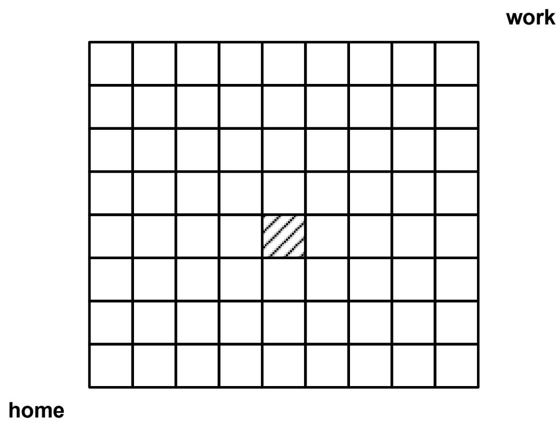
$$a_{17,0} \;\; a_{16,1} \;\; \cdots \;\; a_{10,7} \;\; a_{9,8} \;\; a_{8,9} \;\; a_{7,10} \;\; \cdots \;\; a_{1,16} \;\; a_{0,17}$$

This corresponds to the combination expressions:

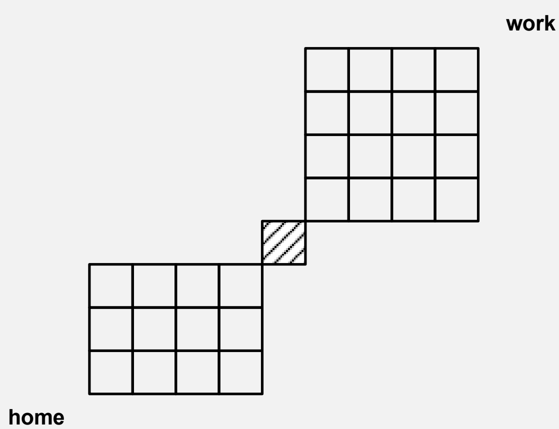
$$\binom{17}{0} \binom{17}{1} \cdots \binom{17}{7} \binom{17}{8} \binom{17}{9} \binom{17}{10} \cdots \binom{17}{16} \binom{17}{17}$$

Therefore the total possible routes for this person to go from home to work is $a_{9,8} = \binom{17}{8} = 24310$.

b. How many different routes are possible if the one block in the easterly direction, which begins four blocks east and three blocks north of his home, is under water (and he can't swim)?



Answer: According the Addition Principle, the number of possible routes that avoid the shaded block can be calculated by the number of total routes with no constraints subtract the number of routs that take the shaded block. Therefore, I would count the routs that use the shaded block. In this case, the map looks like,



The number of possible ways from home to work passing the shaded block is,

$$a_{4,3} \times a_{4,4} = \binom{7}{3} \times \binom{8}{4} = 2450$$

The number of possible ways from home to work without passing the shaded block is,

$$\binom{17}{8} - \left(\binom{7}{3} \times \binom{8}{4}\right) = 24310 - 2450 = 21860\,.$$

Section 5.7

Exercise 7-a: Use the binomial theorem to prove that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Answer: According to binomial theorem, $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$, plug in $x=2,\ y=1$, we have,

$$3^n = (2+1)^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}2^k 1^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}2^k$$

$$\Rightarrow 3^n = \sum_{k=0}^{n}\binom{n}{k}2^k .$$

b. Generalize to fine the sum

$$\sum_{k=0}^{n}\binom{n}{k}r^k$$

for any real number r.

Answer: Following the same logic, we have the relation below,

$$(r+1)^n = \sum_{k=0}^{n}\binom{n}{k}r^k 1^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}r^k$$

$$\Rightarrow (r+1)^n = \sum_{k=0}^{n}\binom{n}{k}r^k .$$

Exercise 16: By integrating the binomial expansion, prove that, for a positive integer n

$$1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$$

Answer: The general form for the left hand side of the equation above is $\frac{1}{k+1}\binom{n}{k}$, where $0 \le k \le n$. We can further convert the this general form as below,

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{k+1}\times\frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k+1)!\,(n-k)!}$$

$$= \frac{(n+1)!}{(k+1)!\,(n-k)!}\times\frac{1}{n+1}$$

$$= \frac{1}{n+1}\binom{n+1}{k+1}$$

We can therefore rewrite the equation to be proven as,

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1} = \frac{2^{n+1}-1}{n+1}$$

$$\Rightarrow \sum_{k=0}^{n} \binom{n+1}{k+1} = 2^{n+1} - 1$$

$$\Rightarrow \sum_{k=1}^{n+1} \binom{n+1}{k} = 2^{n+1} - 1$$

$$\Rightarrow \sum_{k=1}^{n} \binom{n}{k} = 2^{n} - 1 \qquad (to\ be\ proven)$$

We already know from binomial theorem we have the following relation,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\Rightarrow \binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k} = 2^n$$

$$\Rightarrow 1 + \sum_{k=1}^{n} \binom{n}{k} = 2^n$$

$$\Rightarrow \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1\,. \qquad (proved)$$

Exercise 25: Use a combinatorial argument to prove the *Vandermonde* convolution for the binomial coefficients: For all positive integers m_1, m_2, and n

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1 + m_1}{n}$$

Answer: $\binom{m_1}{k}$ can be interpreted as ways of selecting k items from a total of m_1 items with no constraints. $\binom{m_2}{n-k}$ can be interpreted as ways of selecting $n-k$ items from a total of m_2 items with no constraints. So, $\binom{m_1}{k}\binom{m_2}{n-k}$ can be interpreted as ways of the following procedure consisting two steps can be done. The first step is selecting k items from a total of m_1 items. The second step is selecting $n-k$ items from a total of m_2 items. These two steps are independent from each other. Therefore, $\sum_{k=0}^{n}\binom{m_1}{k}\binom{m_2}{n-k}$ can be interpreted as ways of selecting n items from $m_1 + m_2$ items, some of them are from m_1 items the rest of them are from m_2 items. And the procedure should cover all possible ways of how many items each pile contributes. This is same as selecting n items from $m_1 + m_2$ items without any constraints, which is the right hand side.

Exercise 1: Let S be an n-element set and let

$$R = \{(A, B) : A \subseteq B\}$$

That is R is the order relation $\subseteq$ on S. Show that $|R| = 3^n$.

Answer: Assuming I've selected k elements out of n elements from set S to form the set A, then set B can be constructed by taking every element in A and some or nothing from the rest $n-k$ elements. Therefore, given set $|A| = k$, the number of ways to form set B is the number of ways to form a subset from $n-k$ elements. And this number is 2^{n-k}.

Without the assumption, we randomly form set A with k elements. The number of ways to form set A is just $\binom{n}{k}$. Therefore we have the following relations,

$\binom{n}{k}$	Ways to form set A, where $\|A\| = k$
2^{n-k}	Ways to from set B, where $\|A\| = k$
$\binom{n}{k}2^{n-k}$	Ways to from set A and B, where $\|A\| = k$
$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}$	Ways to from set A and B, where $0 \leq \|A\| \leq n$

Therefore, the number of ways to form the relation R is,

$$|R| = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}1^k 2^{n-k}$$

$$= (1+2)^n$$

$$= 3^n .$$

Exercise 2: For an ordinary poker (5-card) hand, find the probability of,

a. The hand has 2 pairs (but not 4 of a kind and not a full house).

Answer:

Ways of selecting two kinds	$\binom{13}{2}$
Ways of selecting two suits from a kind	$\binom{4}{2}$
Ways of selecting the other one card	$\binom{52-4\times2}{1}$
Therefore the answer is	$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{52-4\times2}{1} = 123552$.

b. The hand is a full house

Answer:	Ways of selecting and ordering two kinds	$P(13,2)$
	Ways of selecting three suits from a kind	$\binom{4}{3}$
	Ways of selecting two suits from a kind	$\binom{4}{2}$
	Therefore the answer is	$P(13,2)\binom{4}{3}\binom{4}{2} = 3744$.