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Math 475

Homework #5

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Course site Exercises 4

Problem 1: The procedure below takes an array of integers and determines if some elements occur three (or more) times in the array. Which of the following big-O estimates: $O(\log n)$, $O(n)$, $O(n\log n)$, $O(n^2)$, $O(n^2\log n)$, $O(n^3)$, $O(n^3\log n)$, $O(n^4)$, and $O(2^n)$ best describes the worst-case running time of the algorithm.

```
public boolean hasThreeEqual(int[] arr) {
    int n = array.length;
    for (int i = 0; i < n; i++) {
        for (int j = i + 1; j < n; j++) {
            if (arr[i] == arr[j]) {
                for (int k = j + 1; k < n; k++) {
                    if (arr[i] == arr[k]) {
                        return true;
                    }
                }
            }
        }
    }
    return false;
}
```

Answer: The big-O runtime complexity of this procedure is $O(n^2)$. For the worst case, there shouldn't be any same threes in the middle. If so, function will return without executing the rest of the loops. The elements shouldn't be all different either. If so the most inner loop won't execute. So there are pairs of same elements in the worst case. In this case, the inner loop gets to execute *once* throughout one round of the middle loop execution. An example of such array is $1, 1, 2, 2, 3, 3, \cdots$

$$\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n-1}1+\sum_{k=j+1}^{n-1}1\right)\le\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n-1}1+\sum_{k=i+2}^{n-1}1\right)$$

$$=\sum_{i=0}^{n-1}[(n-i-1)+(n-i-2)]$$

$$=(2n-3)\sum_{i=0}^{n-1}1-2\sum_{i=0}^{n-1}i$$

$$=n^2-2n\,.$$

Therefore the runtime complexity of the given procedure is $O(n^2 - 2n) = O(n^2)$, as we drop the lower order terms and the constants.

Problem 2: Show $B(n) \leq n!$

Answer: We will prove $B(n) \leq n!$ by induction using the recursive relation of Bell numbers,

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$$

Base cases: Recall the definition of Bell numbers, $B(n)$, the number of ways to partition n items. So we have the following simple cases.

$$B(0) = 1 \leq 0! = 1$$

$$B(1) = 1 \leq 1! = 1$$

$$B(2) = 1 + 1 = 2 \leq 2! = 2$$

$$B(3) = 1 + \binom{3}{2} + 1 = 5 \leq 3! = 6\,.$$

Inductive cases: We assume that $B(n) \leq n!$ holds for n, where $n > 3$. Based on the assumption, we need to prove, $B(n+1) \leq (n+1)!$ holds as well, where $n > 3$.

$$\begin{aligned}
B(n+1) &= \sum_{i=0}^{n} \binom{n}{i} B(i) \\
&= \sum_{i=0}^{n-1} \binom{n}{i} B(i) + B(n) \\
&= \sum_{i=0}^{n-1} \frac{n}{n-i} \times \binom{n-1}{i} B(i) + B(n) \\
&\leq \sum_{i=0}^{n-1} \frac{n}{n-(n-1)} \times \binom{n-1}{i} B(i) + B(n) \\
&= n \sum_{i=0}^{n-1} \binom{n-1}{i} B(i) + B(n) \\
&= nB(n) + B(n) \\
&= (n+1)B(n) \\
&\leq (n+1)n! \\
&= (n+1)!\,.
\end{aligned}$$

Therefore we've shown that if $B(n) \leq n!$ holds for n, it also holds for $n+1$. In addition, we also have the base cases proved. Hence we've shown that the expression $B(n) \leq n!$ holds for all $n \geq 0$.

Problem 3: Let V be a vector space of dimension 4 over a finite field with q elements and let $L = Sub(V)$ be the lattice of subspaces. Find $n = |L|$ and the number $e_{\lessdot} = e_{\lessdot}(L)$ of covers in this lattices. If you express $e_{\lessdot}$ as powers of n and take the limit as q goes to infinity, it has the form cn^r plus lower order terms. Find c and r.

Answer: We will use the q-nominal coefficients formula to compute the number of elements in the vector space lattice. The vector space dimension is 4, so we have.

$$
\begin{aligned}
n &= |L| \\
&= \binom{4}{0}_q + \binom{4}{1}_q + \binom{4}{2}_q + \binom{4}{3}_q + \binom{4}{4}_q \\
&= \frac{(4!)_q}{(0!)_q(4!)_q} + \frac{(4!)_q}{(1!)_q(3!)_q} + \frac{(4!)_q}{(2!)_q(2!)_q} + \frac{(4!)_q}{(3!)_q(1!)_q} + \frac{(4!)_q}{(4!)_q(0!)_q} \\
&= 2 \times \frac{(4!)_q}{(0!)_q(4!)_q} + 2 \times \frac{(4!)_q}{(1!)_q(3!)_q} + \frac{(4!)_q}{(2!)_q(2!)_q} = 2 + 2 \times 4_q + \frac{4_q 3_q}{2_q} \\
&= 2 + 2 \times (q^3 + q^2 + q + 1) + \frac{(q^3 + q^2 + q + 1)(q^2 + q + 1)}{(q + 1)} \\
&= 2 + \frac{(q^3 + q^2 + q + 1)[(q^2 + q + 1) + 2(q + 1)]}{(q + 1)} \\
&= 2 + \frac{(q^3 + q^2 + q + 1)(q^2 + 3q + 3)}{(q + 1)}.
\end{aligned}
$$

We will calculate $e_{\lessdot} = e_{\lessdot}(L)$ by counting the number of lines on each level of the vector space lattice, from the dimension above to the one below.

$$
\begin{aligned}
e_{\lessdot} &= e_{\lessdot}(L) \\
&= \binom{4}{1}_q \binom{1}{0}_q + \binom{4}{2}_q \binom{2}{1}_q + \binom{4}{3}_q \binom{3}{2}_q + \binom{4}{4}_q \binom{4}{3}_q \\
&= \frac{(4!)_q}{(1!)_q(3!)_q} \times \frac{(1!)_q}{(0!)_q(1!)_q} + \frac{(4!)_q}{(2!)_q(2!)_q} \times \frac{(2!)_q}{(1!)_q(1!)_q} \\
&= + \frac{(4!)_q}{(3!)_q(1!)_q} \times \frac{(3!)_q}{(2!)_q(1!)_q} + \frac{(4!)_q}{(4!)_q(0!)_q} \times \frac{(4!)_q}{(3!)_q(1!)_q} \\
&= (q^3 + q^2 + q + 1) + \frac{(q^3 + q^2 + q + 1)(q^2 + q + 1)}{(q + 1)} \times (q + 1)
\end{aligned}
$$

$$= +(q^3 + q^2 + q + 1) \times (q^2 + q + 1) + (q^3 + q^2 + q + 1)$$

$$= 2 \times (q^3 + q^2 + q + 1) + 2 \times (q^3 + q^2 + q + 1)(q^2 + q + 1)$$

$$= 2 \times (q^3 + q^2 + q + 1)(q^2 + q + 2)\,.$$

Now we have $n = q^4 + \cdots$, we also have $e_{\prec} = 2q^5 + \cdots$. So, if we express $e_{\prec}$ in the form of n, The n term needs to rise to the power of $\frac{5}{4}$. The coefficient can be computed as below,

$$\sqrt[4]{\frac{e_{\prec}^4}{n^5}} = \sqrt[4]{\frac{2^4 \times q^{20} + \cdots}{q^{20} + \cdots}} = 2 \times \cdots$$

Therefore, if we express $e_{\prec}$ as powers of n and take the limit as q goes to infinity, (in other words, we drop the lower order terms), we have $e_{\prec} = cn^r$, where $c = 2$, and $r = \frac{5}{4}$.

$$e_{\prec} = 2 \times n^{\frac{5}{4}}, \;\; p \to \infty$$

Problem 4: An $n \times n$ matrix is *doubly stochastic* if $0 \leq a_{ij} \leq 1$ and each row sum and each column sum is 1. Prove that if A is doubly stochastic then it has a diagonal all of whose entries are nonzero.

Answer: We will prove the given statement using Frobenius-König theorem. Frobenius-König theorem states a $n \times n$ matrix A, which contains a zero in every one of its diagonals, has a zero sub-matrix of size $s \times t$, where $s + t = n + 1$.

<u>*Assumption:*</u> Let's assume there is a doubly stochastic matrix that has no diagonal containing all nonzero entries. In other words, we assume every diagonal of this doubly stochastic matrix has a zero entry. Therefore, we can apply Frobenius-König theorem on this matrix. This matrix, therefore, has a $s \times t$ zero sub-matrix where $s + t = n + 1$.

<u>*Contradiction:*</u> Now, let's just look at the $s \times t$ zero sub-matrix. Since every row sum and every column sum of a doubly stochastic matrix is 1. The rows and the columns that this $s \times t$ sub-matrix resides would contribute $(s + t) \times 1 = (n + 1) \times 1 = n + 1$ that much to the total entry summation of this matrix. However, the summation of all entries in this $n \times n$ doubly stochastic matrix is only n. We have $partial\ sum = n + 1 > total\ sum = n$, this is a conflict.

<u>*Conclusion:*</u> the assumption, "there is a doubly stochastic matrix that has no diagonal containing all nonzero entries", was incorrect. Hence, we've shown a doubly stochastic matrix has a diagonal all of whose entries are nonzero.

Example: If we have a 4×4 doubly stochastic matrix, and it has no diagonal containing all nonzero entries. We apply Frobenius-König theorem on it. After some row exchanges and some column exchanges, we would get a $s \times t$ zero sub-matrix, where $s + t = 4 + 1 = 5$. Let's just say the 3×2 sub-matrix on the top left corner is a zero sub-matrix. In other words, we now have three rows and two columns intersect into zeros.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\Rightarrow \begin{cases} a_{13} + a_{14} = 1 \\ a_{23} + a_{24} = 1 \\ a_{23} + a_{34} = 1 \\ a_{41} = 1 \\ a_{42} = 1 \end{cases}$$

$$\Rightarrow a_{13} + a_{14} + a_{23} + a_{24} + a_{23} + a_{34} + a_{41} + a_{42} = 5$$

However, according to the definition of doubly stochastic matrix, we have

$$a_{11} + a_{12} + a_{13} + a_{14} = 1$$
$$a_{21} + a_{22} + a_{23} + a_{24} = 1$$
$$a_{31} + a_{32} + a_{33} + a_{34} = 1$$
$$a_{41} + a_{42} + a_{43} + a_{44} = 1$$

$$\Rightarrow a_{11} + \cdots + a_{14} + a_{21} + \cdots + a_{24} + a_{31} + \cdots + a_{34} + a_{41} + \cdots + a_{44} = 4$$

The total sum can't be smaller than a partial sum. Therefore, this is a contradiction.

Section 8.6

Exercise 12: Prove that the Stirling numbers of the second kind satisfy the following relations:

a. $S(n, 1) = 1, \ (n \geq 1)$

Answer: Recall the definition of Stirling numbers of the second kind, $S(n, k)$ is the number of ways to partition n items to k nonempty blocks. There is only one way to partition n items into 1 block. Therefore $S(n, 1) = 1$, where $n \geq 1$.

b. $S(n, 2) = 2^{n-1} - 1,\ (n \geq 2)$

Answer: There are 2^n ways to form a subset from a collection of n items, including the empty subset. In the context of partitions, selecting the current set of items is the same as selecting the rest of the items. Therefore, there are 2^{n-1} ways to form 2 partitions if we count empty set. Exclude the one that has the empty subset, we have $2^{n-1} - 1$ ways to form 2 partitions from n items. In other words, $S(n, 2) = 2^{n-1} - 1$.

c. $S(n, n-1) = \binom{n}{2},\ (n \geq 2)$

Answer: To form $n - 1$ partitions from n items, we simply choose2 items to form one partition and place the rest of them, the $n - 2$ items, in their own partitions, $n - 2$ partitions. There are $\binom{n}{2}$ ways to select the two items that go together, so there are $\binom{n}{2}$ ways to form $n - 1$ partitions from n items. In other words, $S(n, n-1) = \binom{n}{2}$.

d. $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4},\ (n \geq 2)$

Answer: We will prove $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ where $n \geq 2$, by induction based on the recurrence relation of the Stirling numbers of the second kind, $S(n, k) = kS(n-1, k) + S(n-1, k-1)$.

Base case: $S(2, 0)$ is defined to be 0. $S(3, 1)$ is the number of ways to partition 3 items into 1 block. There's only one way to do so.

$$S(2, 0) = 0 = \binom{2}{3} + 3\binom{2}{4} = 0 \qquad \text{Note } \binom{n}{k} = 0, \text{ where } k > n$$

$$S(3, 1) = 1 = \binom{3}{3} + 3\binom{3}{4} = 1 \qquad \text{Note } \binom{n}{k} = 0, \text{ where } k > n$$

Inductive cases: We assume that $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ holds for n, where $n \geq 2$. We will now prove $S(n+1, n-1) = \binom{n+1}{3} + 3\binom{n+1}{4}$ also holds, where $n \geq 2$.

$$S(n, k) = kS(n-1, k) + S(n-1, k-1) \Rightarrow S(n+1, n-1)$$

$$= (n-1)S(n, n-1) + S(n, n-2)\,.$$

Using the recursive relation, and the conclusion from the previous exercise, $S(n, n-1) = \binom{n}{2}$, where $n \geq 2$ we can derive the following equation.

$$S(n+1, n-1) = (n-1)S(n, n-1) + S(n, n-2)$$

$$= (n-1) \times \binom{n}{2} + \binom{n}{3} + 3\binom{n}{4}$$

$$= \frac{(n-1) \times n!}{2! \times (n-2)!} + \frac{n!}{3! \times (n-3)!} + \frac{3 \times n!}{4! \times (n-4)!}$$

$$= \frac{1}{24} n(n-1)\,(n+1)(3\,n-2)\,.$$

We now simplify the right side of the to be proven equation,

$$\binom{n+1}{3} + 3\binom{n+1}{4} = \frac{(n+1)!}{3!\,(n-2)!} + \frac{3 \times (n+1)!}{4!\,(n-3)!}$$

$$= \frac{1}{24} n(n-1)\,(n+1)(3\,n-2)\,.$$

So, we've shown $S(n+1, n-1) = \binom{n+1}{3} + 3\binom{n+1}{4}$ holds if $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ holds. Plus the base cases, we've proven that $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ holds for all $n \geq 2$.

Exercise 19: Prove that the Stirling numbers of the first kind satisfy the following formulas:

a. $s(n, 1) = (n-1)!$, $(n \geq 1)$

Answer: Recall the definition of Stirling numbers of the first kind, $s(n, k)$ is the number of ways to partition n items to k circular groups

There are $n!$ permutations of n items. Every n permutations fall into a same circular group. For example, we have a set with four items, $\{a, b, c, d\}$. The following four permutations fall into the same circular group, $\{a, b, c, d\}, \{b, c, d, a\}, \{c, d, a, b\}$ and $\{d, a, b, c\}$.

Therefore there are $\frac{n!}{n} = (n-1)!$ ways to partition n items into 1 circular group, where $n \geq 1$.

b. $s(n, n-1) = \binom{n}{2}$, $(n \geq 1)$

Answer: The argument is quite similar to the argument for $S(n, n-1) = \binom{n}{2}$. To form $n-1$ circular groups from n items, we simply choose2 items to form one circular group ($\{a, b\}, \{b, a\}$ are in the same circular group), and then we place the rest of items, the $n-2$ of them, in their own groups,

$n-2$ groups. There are $\binom{n}{2}$ ways to select the two items that go together, so there are $\binom{n}{2}$ ways to form $n-1$ circular groups from n items. In other words, $s(n, n-1) = \binom{n}{2}$.