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Math 475

Homework #6 (Final)

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Course site Final Exercises

Problem 1: Recall that the order dimension of a partially ordered set is the smallest number of linear extensions that intersect to it. Find the order dimension of the partially ordered set below (and show your answer is correct, of course). Note c and d are incomparable.

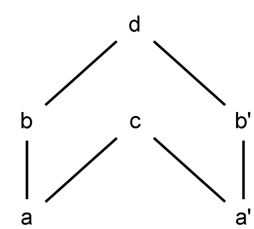


Figure 1

Answer: It is not hard to find a set of three linear extensions that intersect to the partial order set above. For example, the following three:

$$\{a, b, a', c, b', d\}$$

$$\{a', b', a, b, d, c\}$$

$$\{a, a', c, b, b', d\}$$

There are 7 incomparable pairs of elements in the given partial order set. They are $(a, a'), (a, b'), (b, a'), (b, c), (b, b'), (c, b')$, and (c, d). Each of the linear extension listed above contains the following ordering relations for the incomparable element pairs:

$$(a \prec a'), (a \prec b'), (b \prec a'), (b \prec c), (b \prec b'), (c \prec b'), (c \prec d) \in \{a, b, a', c, b', d\}$$

$$(a \succ a'), (a \succ b'), (b \succ a'), (b \prec c), (b \succ b'), (c \succ b'), (c \succ d) \in \{a', b', a, b, d, c\}$$

$$(a \prec a'), (a \prec b'), (b \succ a'), (b \succ c), (b \prec b'), (c \prec b'), (c \prec d) \in \{a, a', c, b, b', d\}$$

As we can see, all 7 incomparable pairs are listed in both directions in these three linear extensions. The ordering relation of these incomparable pairs, therefore, will not appear in the intersection of these linear extensions.

Now we know that 3 linear extensions are definitely enough to recover the original partial order relation. We need to decide whether 3 is the smallest number of linear extensions that be able to accomplish this task. If it is, by definition, the order dimension of the given partial order set is 3. So, we will attempt to find a set of 2 linear extensions that intersect into the original partial order set. If we can prove that 2 is not possible, then we know 3 must be the answer.

Assume we have 2 linear extensions that intersect into the given partial order relation. We will call them le_1 and le_2. Since c and d are incomparable, $(c \prec d)$ and $(d \prec c)$ need to appear in le_1 and le_2 separately. WLOG, we have the following:

$(c \prec d) \in le_1$

$(d \prec c) \in le_2$

$\Rightarrow (d \prec c), (b \prec c), (b' \prec c) \in le_2 \quad since\ (b \prec d)\ and\ (b' \prec d)$

$\Rightarrow (c \prec d), (c \prec b), (c \prec b') \in le_1 \quad since\ (b \prec c), (b' \prec c) \in le_2$

$\Rightarrow (c \prec d), (c \prec b), (c \prec b'), (a \prec b'), (a' \prec b) \in le_1 \quad since\ (a \prec c)\ and\ (a' \prec c)$

$\Rightarrow (d \prec c), (b \prec c), (b' \prec c), (b' \prec a), (b \prec a') \in le_2 \quad since\ (a \prec b'), (a' \prec b) \in le_1$

Now we have a conflict in le_2. If $(b' \prec a)$ and $(b \prec a')$ are both true in one linear extension, this linear extension cannot start with either a or a'. This is a contradiction. a and a' are the two minimal elements of the given partial order set. One of them needs to appear as the first element in the linear extension of this set.

Therefore, we've shown a set of linear extensions with 2 elements cannot intersect into the given partial order relation. We've also shown, with an example, that a set of linear extensions with 3 elements can intersect into the given partial order set. We conclude that the order dimension of this partial order set is 3.

Problem 2: Let $S_1, \cdots, S_n$ be subsets of $\{x_1, \cdots, x_n\}$. Suppose $|S_i| \geq k, i = 1, \cdots, n$ and that each x_j lies in at least k of the subsets.

a. Show that there are at least $k!$ SDR's for this system

Answer: We will prove this statement by induction. First we'll show a couple of <u>*base cases*</u>. When $n = 2, k = 1$, there's $1! = 1$ SDR.

$$\{x_1, \cdots, x_n\} = \{1, 2\}$$

$$\{S_1, \cdots, S_n\} = \{\{1\}, \{2\}\}$$

$$1\ SDR\ assignment: 1 \rightarrow \{1\}, 2 \rightarrow \{2\}$$

When $n = 3, k = 1$, there's $1! = 1$ SDR.

$$\{x_1, x_2, \cdots, x_n\} = \{1, 2, 3\}$$

$$\{S_1, S_2, \cdots, S_n\} = \{\{1\}, \{2\}, \{3\}\}$$

$$1\ SDR\ assignments: 1 \to \{1\}, 2 \to \{2\}, 3 \to \{3\}$$

When $n = 3, k = 2$, there're $2! = 2$ SDRs.

$$\{x_1, x_2, \cdots, x_n\} = \{1, 2, 3\}$$

$$\{S_1, S_2, \cdots, S_n\} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

$$2\ SDR\ assignments: 1 \to \{1, 2\}, 2 \to \{2, 3\}, 3 \to \{1, 3\}$$

$$1 \to \{1, 3\}, 2 \to \{1, 2\}, 3 \to \{2, 3\}$$

Now we will show the *inductive case*. In other words, we need to show if the given statement holds for $n - 1$ and $k - 1$, it's enough to derive that it also holds for n and k.

We have $S_1, \cdots, S_n$ being subsets of $\{x_1, \cdots, x_n\}$, and x_i lies in at least k subsets of S_i. We can choose any element x from S_n as its representative, and there are at least k choices for x, since $|S_n| \geq k$. Now we remove x from $\{x_1, \cdots, x_n\}$, and remove S_n from $S_1, \cdots, S_n$. We also remove x from S_i, if $x \in S_i$. Let's call the newly formed subsets $S'_1, \cdots S'_{n-1}$. Then we are left with $X' = \{x' : x' \in \{x_1, \cdots, x_n\}\ and\ x' \neq x\}$, and $S'_1, \cdots S'_{n-1}$, where x'_i lies in at least $k - 1$ subsets of S'_i, since we only removed S_n. There are $|X'| = n - 1$ elements, and $n - 1$ sets, with $k - 1$ coverage. The problem is converted to $n - 1$ and $k - 1$. If the number of SDRs for $n - 1$ and $k - 1$ is $(k - 1)!$, there are $(k - 1)! \times k = k!$ ways to form SDRs for n and k. Therefore, we've shown if and only if the statement holds for $n - 1$ and $k - 1$, it holds for n and k.

Sum up the base cases and the inductive case, we conclude that there are at least $k!$ SDRs for the system, where $S_1, \cdots, S_n$ being subsets of $\{x_1, \cdots, x_n\}$. and $|S_i| \geq k, i = 1, \cdots, n$ with each x_j lies in at least k of the subsets.

b. An $r \times n$ partial Latin square based on $\{1, \cdots, n\}$ is an $r \times n$ matrix such that each row is a permutation and such that the r elements of each column are distinct. If $r < n$ show that a $r \times n$ partial Latin square can be extended to an $(r + 1) \times n$ partial Latin square in at least $(n - r)!$ ways.

Answer: Let S_j be the set of numbers that do not appear in column j. The $(r + 1)$ row corresponds precisely to a system of distinct representatives for the collection $S_1, \cdots S_n$. Now we verify the Marriage Condition on this system. Every set S_j has size $n - r$, and every element is in precisely $n - r$ sets, since it is already listed r times in the $r \times n$ rectangle. Any m of the sets S_j contain together $m(n-r)$ elements and therefore at least $\frac{m(n-r)}{n-r} = m$ different elements in there. So we've shown that the Marriage Condition holds, and therefore the system has a SDR, which is what we apply to form the $(r+1)$ row.

We also notice that this system satisfy the condition of the previous question, where $S_1, \cdots, S_n$ being subsets of $\{x_1, \cdots, x_n\}$, $|S_i| = n - r, i = 1, \cdots, n$, and each x_j lies in $n - r$ of the subsets. We, therefore, apply the conclusion of the previous question, and derive that there are at least $(n-r)!$ SDRs for to form the $(r+1)$ row. Hence, we've shown a $r \times n$ partial Latin square can be extended to an $(r+1) \times n$ partial Latin square in at least $(n-r)!$ ways.

c. Exhibit two 2×4 partial Latin squares based on $\{1,2,3,4\}$, both with first row $[1,2,3,4]$, such that the number of ways of adding a third row is different.

Answer: We've shown that the lower bound of ways to extend a $r \times n$ partial Latin square to a $(r+1) \times n$ partial Latin square. There are, therefore, at least $(4-2)! = 2$ ways to extend a 2×4 partial Latin square to a 3×4 partial Latin square. But the actual number might be greater, just like we've proven in class the Derangement, which is the way of forming the second row, is greater than this lower bound.

$$\frac{n!}{e} > \frac{n!}{n} = (n-1)!$$

The following two 2×4 partial latin squares based on $\{1, 2, 3, 4\}$ have different numbers of ways of forming the third row.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} \quad vs \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Set $\{1,2,3,4\}$ is small enough of a set that we are able to list out all the possible row permutations, $4! = 24$ of them, and look through the list to find valid third rows.

$[1,2,3,4]$	$[2,1,3,4]$	$[3,1,2,4]$	$[4,1,2,3]$
$[1,2,4,3]$	$[2,1,4,3]$	$[3,1,4,2]$	$[4,1,3,2]$
$[1,3,2,4]$	$[2,3,1,4]$	$[3,2,1,4]$	$[4,2,1,3]$
$[1,3,4,2]$	$[2,3,4,1]$	$[3,2,4,1]$	$[4,2,3,1]$
$[1,4,2,3]$	$[2,4,3,1]$	$[3,4,1,2]$	$[4,3,1,2]$
$[1,4,3,2]$	$[2,4,1,3]$	$[3,4,2,1]$	$[4,3,1,2]$

Looking through the list, there are 2 ways to form the third row for partial Latin square A. They are $[3,4,1,2]$ and $[4,1,2,3]$. There are 4 ways to form the third row for partial Latin square B. They are $[2,1,4,3]$, $[2,3,4,1]$, $[4,1,2,3]$ and $[4,3,2,1]$.

Problem 3: How many ways can 8 pieces of distinguishable candies be distributed to 4 (distinguishable) children so that each child gets at least one piece?

Answer: According to the onto-function formula, the number of onto functions from a set containing 8 elements to a set containing 4 elements, without empty sets, is,

$$\begin{aligned} 4^8 - \binom{4}{3}3^8 + \binom{4}{2}2^8 - \binom{4}{1}1^7 &= 65536 - 4 \times 6561 - 6 \times 256 - 4 \times 1 \\ &= 65536 - 26244 + 1536 - 4 \\ &= 40824\,. \end{aligned}$$

Another approach is starting from the Stirling numbers of the second kind. By definition, the Stirling number of the second kind, $S(n,k)$, is the ways to partition a set of n elements into k nonempty subsets. Now, k partitions are assigned to distinguishable objects. Therefore we need to permute k as well. Hence, we have,

$$S(8,4) \times 4! = 1701 \times 24 = 40824\,.$$

The Stirling number of the second kind can be computed using its recursive relation, $S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$. The steps of computing $S(8,4)$ are omitted here.

There are 40824 ways to distribute 8 pieces of distinguishable candies to 4 distinguishable children so that each child gets at least one piece.

Problem 4: How many ways can 8 pieces of distinguishable candies be distributed to 4 indistinguishable bags so that the following conditions hold.

a. Each bag has at least one piece.

Answer: We need to partition 8 distinguishable objects into 4 indistinguishable block without empty sets. This is precisely the Stirling number of the second kind. Hence, we have, $S(8,4) = 1701$, ways of distributing 8 pieces of distinguishable candies into 4 indistinguishable bags where each bad has at least one piece.

b. Some bags can be empty.

Answer: We will apply the Stirling numbers and count the number of empty bags one by one.

There are no empty bags:	$S(8,4) = 1701$
There is one empty bag:	$S(8,3) = 966$
There are two empty bags:	$S(8,2) = 127$
There are three empty bags:	$S(8,1) = 1$

The total number of ways to distribute the candies is therefore, $1701 + 966 + 127 + 1 = 2795$.

Problem 5: A projective plane is a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines and an incidence relation $I \subseteq \mathcal{P} \times \mathcal{L}$ which tells you if "p is on L", for $p \in \mathcal{P}$ and $L \in \mathcal{L}$. It's ok to think of each line as just the set of point incident with it and we will do that. The axioms are

1. Any two distinct points lie on exactly one line.
2. Any two distinct lines have exactly one point incident with both.
3. There are 4 points, no three of which lie on a line.

a. Suppose L_1 and L_2 are lines and p is a point not on either. (p exists by the third axiom.) Let $f: L_1 \to L_2$ be given by

$$f(x) = (x \vee p) \wedge L_2$$

where $x \vee p$ is the line containing both x and p and $(x \vee p) \wedge L_2$ is the common point of those 2 lines. Show that f is one-to-one and onto.

Answer: *Proof of one-to-one:* A function $g: A \to B$ is called a one-to-one function, if and only if, for all elements x, y in A, if $g(x) = g(y)$, then $x = y$. Therefore, in order to prove $f(x) = (x \vee p) \wedge L_2$ is a one-to-one function, we need to assume $f(x_1) = f(x_2)$, and show $x_1 = x_2$.

Since $f(x)$ indicates a point on L_2. $f(x_1)$ and $f(x_2)$, where $f(x_1) = f(x_2)$, indicate a same point L_2. Let's call this point y. According to the definition of function f, we have the following relation,

$$(x_1 \vee p) \wedge L_2 = y \Rightarrow x_1 = (y \vee p) \wedge L_1$$

$$(x_2 \vee p) \wedge L_2 = y \Rightarrow x_2 = (y \vee p) \wedge L_1$$

According to the second axiom, any two distinct lines have exactly one point incident with both. Since $p \notin L_1$, line $y \vee p$ and line L_1 are two distinct lines. There is only one point they incidence with each other. Therefore, $x_1 = x_2$. Hence, we've shown if $f(x_1) = f(x_2)$, it follows $x_1 = x_2$. Function $f(x) = (x \vee p) \wedge L_2$ is a one-to-one function.

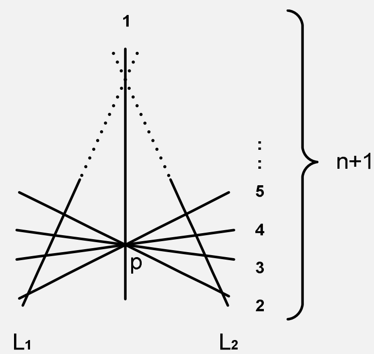
Proof of onto: A function $g: A \to B$ is called a onto function, if and only if, for all elements in B, we can find some elements in A with the property that $y = g(x)$, where $y \in B$ and $x \in A$. Therefore, in order to prove $f(x) = (x \vee p) \wedge L_2$ is a onto function, we need to assume $y \in L_2$ and show $\exists x \in L_1$ such that $f(x) = y$.

Let's form a line using y and p, $y \vee p$. Since $p \notin L_1$, line $y \vee p$ and line L_1 are two distinct lines. According to the second axiom, any two distinct lines have one and exactly one point incident with both. Line $y \vee p$ and line L_1 have to incidence. Let's call this point x, where $x \in L_1$. So, we have $(x \vee p) \wedge L_2 = y$ and $f(x) = y$. Hence we've show if $y \in L_2$, $\exists x \in L_1$ such that $f(x) = y$. Function $f(x) = (x \vee p) \wedge L_2$ is a onto function.

b. Suppose $\mathcal{P}$ and $\mathcal{L}$ are finite. Show that every line has the same number of points on it. Let $n + 1$ be this common number. Show that every point has exactly $n + 1$ lines going through it.

Answer: We've proven there is a one-to-one and onto mapping from L_1 to L_2, where L_1 and L_2 are any two lines of the projective plane. We can apply the property of the bijective function (both one-to-one and onto). If X and Y are finite sets, then there exists a bijection between the two sets X and Y if and only if X and Y have the same number of elements, $|X| = |Y|$. The cardinalities of L_1 and L_2, therefore, have to be the same, $|L_1| = |L_2|$. The same rule applies to the rest of the lines, since L_1 and L_2 were randomly selected, $|L_i| = |L_j|$.

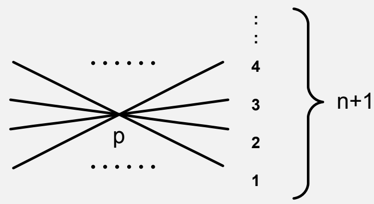
Let $n + 1$ be the common number of points on a line, $|L_1| = |L_2| = n + 1$. Considering the point p in function $f(x) = (x \vee p) \wedge L_2$, there are *at least* $n + 1$ lines going through p from L_1 and L_2, since $|L_1| = |L_2| = n + 1$.



Also, there are *exactly* $n + 1$ lines through p. Assume there is a line l such that $p \in l$, and $l \wedge L_1 = \emptyset$. Since $p \notin L_1$, l and L_1 are distinct, according the second axiom, two distinct lines have to incident. This is a conflict with the assumption that $l \wedge L_1 = \emptyset$. The assumption, therefore, does not hold. No such line l exists. There are, therefore, precisely $n + 1$ lines crossing p. Since p is a randomly selected point from the project plane, we've shown every point has exactly $n + 1$ lines going through it.

c. Find $|\mathcal{P}|$ and show it is the same as $|\mathcal{L}|$.

Answer: We will follow the conventions from the previous question. There are $n + 1$ lines crossing every point and $n + 1$ points on each line. We will start with considering a point p and find out the total number points in a project plan, $|\mathcal{P}|$.



There are $n+1$ lines crossing p, and each line contains $(n+1)-1=n$ points other than p. We can prove these points are all distinctive from each other. Assume there is a point $p' \in L_1$, and also $p' \in L_2$, where $L_1 \neq L_2$ and $L_1 \wedge L_2 = p$. We would have $L_1 \wedge L_2 = p, p'$, which is a conflict with the second axiom, that two distinct lines incidence in one and only one point. Therefore, the assumption is wrong. The$(n+1) \times n$ points on these $n+1$ lines are all distinct. So, we learned that there are *at least* $(n+1) \times n + 1 = n^2+n+1$ points in this projective plane, $|\mathcal{P}| \geq n^2+n+1$.

We also need to show there are *exactly* n^2+n+1 points, $|\mathcal{P}| = n^2+n+1$, in other words, no others points that were not counted. Assume there is a point p' that was not counted in the configuration above. Since p' and p are two distinct points, according to the first axiom, any two distinct points lie on exactly one line, there must be a line going across p' and p. This is a contradiction with the assumption that p' was not counted. Therefore, the assumption is wrong. We've counted all points in the project plane, $|\mathcal{P}| = n^2+n+1$.

Sum up the two arguments above, we conclude that there are n^2+n+1 points in the projective plan. We will now derive $|\mathcal{L}|$. There are $|\mathcal{L}|$ lines in the projective plane, each line contains $n+1$ points, and each points is counted $n+1$ times. So the number of points in the project plane can be expressed using $|\mathcal{L}|$ as,

$$\frac{|\mathcal{L}| \times (n+1)}{n+1} = |\mathcal{P}| = n^2+n+1$$

$$\Rightarrow |\mathcal{L}| = |\mathcal{P}| = n^2+n+1\,.$$

d. Show that $\mathcal{L}$ is a SBIBD with parameters $\lambda = 1, k = n+1$, and $v = n^2+n+1$.

Answer: We will prove the statement by showing the BIBD parameters one by one. If it is also a SBIBD, it would become clear once the parameters are all resolved. Let's consider the points as the varieties, the lines as the blocks.

	The number of points	$\|\mathcal{P}\| = n^2+n+1$
$\Rightarrow$	The number of varieties	$v = n^2+n+1$
	The number of lines	$\|\mathcal{L}\| = n^2+n+1$
$\Rightarrow$	The number of blocks	$b = n^2+n+1$
	The number of points in each line	$\|\{p: p \in l, l \in \mathcal{L}, p \in \mathcal{P}\}\| = n+1$

$\Rightarrow$ The number of varieties in each block $k = n + 1$

The number of lines crossing each point $|\{l: p \in l, l \in \mathcal{L}, p \in \mathcal{P}\}| = n + 1$

$\Rightarrow$ The number of blocks containing a fixed variety $r = n + 1$

The number of lines crossing two points $|\{l: p_1 p_2 \in l, l \in \mathcal{L}, p_1 p_2 \in \mathcal{P}\}| = 1$

$\Rightarrow$ The number of blocks containing a fixed pair of varieties $\lambda = 1$

Therefore, we've shown that a projective plane is a BIBD. We also notice that $v = b = n^2 + n + 1$ and $k = r = n + 1$. So a projective plane is a $(n^2 + n + 1, n + 1, 1)$ SBIBD.

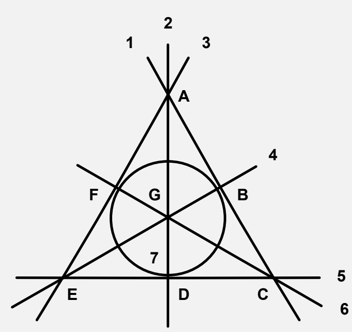
e. Show that $\mathcal{L}$ has an SDR.

Answer: In order to show that $\mathcal{L}$ has a system of distinct representatives, according to the Hall's theorem, we need to show the Marriage Condition holds for the projective plane, where points indicate varieties and lines indicate blocks. In other words, if we pick m lines from a projective plane, we need to show on these m lines there are at least m distinct points.

Number of points in these m lines (including duplicates)	$m \times (n + 1)$
Number of times a point is counted in $m \times (n + 1)$	$\leq n + 1$
Number of different points in these $m \times (n + 1)$ points	$\geq \frac{m \times (n+1)}{n+1} = m$

Therefore, we've shown there are at least m different points in a random selection of m lines in the projective plane. Hence, we've shown that the Marriage Condition holds for a projective plane. The collection of lines, $\mathcal{L}$, therefore, has an SDR.

For example in for following projective plane, we have the following SDR for all of the lines.



$point\ A \rightarrow line\ 1$ $point\ B \rightarrow line\ 3$ $point\ C \rightarrow line\ 6$ $point\ D \rightarrow line\ 7$

$point\ E \rightarrow line\ 4$ $point\ F \rightarrow line\ 2$ $point\ G \rightarrow line\ 5$