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Math 611

Extra Credit for Final

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Extra Credit

Question: Prove that $R[x]$ is a PID if and only if R is a field, where R is a commutative ring with 1.

Answer: We will first prove the "if" direction. If R is a field, we want to show that it implies $R[x]$ is a PID. This conclusion can be derived from Theorem 12, which states, $F[x]$ is a Euclidean Domain (ED) if F is a field. R is indeed a field, so $R[x]$ is a ED. Since ED belongs in PID, so $R[x]$ is a PID.

Now we will prove the "only if" direction. If $R[x]$ is a PID, we want to show that it implies R is a field. Let's consider the ideal generated by two polynomials a and x, $(a, x) \subseteq R[x]$, where $a \in R, a \neq 0$. Since $R[x]$ is a PID, (a, x) must be one generated. Let's call the generating element $f(x)$. So we have $(a, x) = (f(x))$. Hence we have the following relationship:

$$\begin{aligned} &(a,\ x) = (f(x)) \\ &\Rightarrow \exists w(x) \in R[x],\ a = f(x) \cdot w(x). \end{aligned}$$

Case #1:

If R is an Integral Domain (ID), then we can derive that $f(x)$ and $w(x)$ are constant polynomials, since $a = f(x) \cdot w(x)$ and a is a constant polynomial. There's no non-zero zero divisors in an ID, so non-constant polynomials cannot multiply to result in a constant polynomial. Namely, in an ID, $deg(q(x) \cdot p(x)) = deg(q(x)) + deg(p(x))$.

$$\begin{aligned} &deg(f(x) \cdot w(x)) = deg(a) = 0 \\ &\Rightarrow deg(f(x)) + deg(w(x)) = 0 \\ &\because deg(f(x)) \geq 0,\ deg(w(x)) \geq 0 \\ &\Rightarrow deg(f(x)) = deg(w(x)) = 0. \end{aligned}$$

Meanwhile, $f(x)$ generates x. Say $f(x) = f$, then $\exists f^{-1} \in R$ such that $f \cdot f^{-1} x = x$. Therefore $f(x) = f$ is a unit, so $(f(x)) = R$. In other words, $1_{R[x]} \in (f(x))$, so $(f(x)) = (a, x) = R[x]$.

Now we have the following relationship:

$$\begin{aligned}
&1_{R[x]} \in (a,\ x) \\
&\Rightarrow 1_{R[x]} = r \cdot a + t \cdot x \\
&\Rightarrow r = a^{-1},\ t = 0_{R[x]}.
\end{aligned}$$

Since a is randomly selected from R and we've found a multiplicative inverse for a, plus R is an ID, we conclude that R is a field.

Case #2:

If R is not an ID, we will show that there is a conflict. It is easy to see that $deg(f(x) \cdot w(x)) \leq deg(f(x)) + deg(w(x))$. The equal happens when R is an ID, or simply the leading coefficients of $f(x)$ and $w(x)$ are not zero divisors in R.

Let's assume that there exist $\alpha, \beta \in R$ and α, β are non-zero zero divisors. $\alpha \cdot \beta = \beta \cdot \alpha = 0$ and $\alpha \neq 0, \beta \neq 0$. Let's also have the following polynomials:

$$\begin{aligned}
g_1(x) &= \alpha x + \alpha \\
g_2(x) &= \beta x + \beta \\
\because & \quad \alpha \neq 0,\ \beta \neq 0 \\
\Rightarrow & \quad g_1(x) \neq 0_{R[x]},\ g_2(x) \neq 0_{R[x]} \\
g_1(x) \cdot g_2(x) &= \alpha\beta(x+1)^2 = 0_{R[x]}.
\end{aligned}$$

This is a conflict with the fact that $R[x]$ is a PID. $R[x]$ is a PID, so $R[x]$ is a UFD as well as an ID. In other words, if the production of two non-zero polynomials in $R[x]$ is the zero polynomial, then one of them has to be the zero polynomial. But in the example shown above, Neither $g_1(x)$ nor $g_2(x)$ is the zero polynomial, their production is, however, the zero polynomial.

Therefore, the assumption was not correct. R cannot contain non-zero zero divisors α and β. Since R is given a commutative ring and R doesn't have any non-zero zero divisors, R is an ID. The rest of the proof follows as *Case #1*.