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Math 611

Homework #4

November 24, 2010

Exercises in BAI 2.2

Question #1: R is a ring with 1. Show that if $1 - ab$ is a unit, then $1 - ba$ is also a unit.

Answer: Since $1 - ab$ is a unit, there exist $x \in R$, such that $(1 - ab) \cdot x = x \cdot (1 - ab) = 1$. We can show that $1 - ba$ is a unit by proving $(1 + bxa)$ is its inverse. First, $(1 + bxa) \in R$ becuase R is closed under addition and multiplication.

$$\begin{aligned}
(1 - ba) \cdot (1 + bxa) &= 1 + bxa - ba - babxa \\
&= 1 - ba + b(1 - ab)xa \\
&= 1 - ba + b[(1 - ab)x]a \\
&= 1 - ba + ba \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
(1 + bxa) \cdot (1 - ba) &= 1 - ba + bxa - bxaba \\
&= 1 - ba + bx(1 - ab)a \\
&= 1 - ba + b[x(1 - ab)]a \\
&= 1 - ba + ba \\
&= 1.
\end{aligned}$$

In summary, we've shown that $(1-ba)\cdot(1+bxa) = (1+bxa)\cdot(1-ba) = 1$, where $1+bxa \in R$. $1 - ba$ is therefore a unit in R.

Question #4: Show that a finite domain is a skew field.

Answer: Let's call this finite domain F. By the definition of integral domain, if $a, b \in F$ and $ab = 0$ then either $a = 0$ or $b = 0$. The goal for this question is to show that $\forall x \in F$, it has an verse. In other words, $\exists y \in F$, such that $xy = 1$.

Let's define a map, $\varphi : F \to F$, as $\varphi(a) = ab$, where $a, b \in F$, and $b \neq 0$ is an arbitrarily chosen fixed element. We can show that this map is a one-to-one map. Assume $\varphi(x) = \varphi(y)$, then we have:

$$
\begin{aligned}
& xb = yb \\
& \Rightarrow xb - yb = 0 \\
& \Rightarrow (x-y)b = 0 \\
& \Rightarrow x - y = 0 \; or \; b = 0 \\
& \Rightarrow x - y = 0 \\
& \Rightarrow x = y.
\end{aligned}
$$

According to the ***Pigeonhole Principle***, any one-to-one map from a set onto another set with the same finite cardinality, is also an onto map. So φ is also an onto map from F to F. In other words, for any $y \in F$, there exists some $x \in F$ such that $xb = y$. Further, if $y = 1$, then we are saying for a fixed element b, there always exists $xb = 1$. So, we've found a inverse for b. Since b is arbitrarily chosen, we have $\forall b \in F, b \neq 0, \exists x \in F$, such that $xb = 1$.

In summary, we've shown that $\forall a \in F$, a has an inverse, therefore F is a skew field. If F is commutative, then it forms a field.

Exercises in DF 7.4

Question #37: A commutative ring R is called a local ring if it has a unique maximal ideal.

a. Prove that if R is a local ring with the maximmal ideal M, then every element from $R - M$ is a unit.

Answer: We will prove this by controdiction. Let's take an element $x \in R - M$, and assume that x is a nonunit. Let's consider the principle ideal generated by x, (x). By ***Zorn's Lamma***, if (x) is an ideal of R, it has to lay inside of some maximal ideal. Since there is only one maximal ideal, we learn that $(x) \subseteq M$. Therefore, $x \in M$. This is a controdiction of the assumption that $x \in R - M$. So, the assumption doesn't hold. In other words, if an element $x \in R - M$, then it is a unit.

b. Prove conversely that if R is a commutative ring with 1 in which the set of nonunits forms an ideal M, then R is a local ring with unique maximal ideal M.

Answer: We will first show that M is maximal. This is easy. If there is an ideal N, such that $M \subset N$, there exist some element $x \in N - M$. Since M contains all element that are nonunits in R, x must be a unit. In other words $N = R$. So M is a maximal ideal

$$\begin{aligned}
&M \subset N \\
&\Rightarrow \exists x \in N,\ x \notin M \\
&\Rightarrow x\ is\ a\ unit \\
&\Rightarrow N = R.
\end{aligned}$$

We also need to show that M is the only maximal ideal. Let's assume there exists another maximal ideal $O \neq M$.

$$\begin{aligned}
&O\ is\ maximal,\ O \neq M \\
&\Rightarrow \exists y \in O,\ y \notin M \\
&\Rightarrow y\ is\ a\ unit \\
&\Rightarrow O = R.
\end{aligned}$$

So, M is the unique maximal ideal of R. R is a local ring with the maximal ideal M.

Question #38: Prove that the ring of all rational numbers whose denominators is odd is a local ring whose unique maximal ideal is the princeple ideal generated by 2.

Answer: First we will show that the set R of all rational numbers whose denominators is odd form a commutative ring. Let's have $x = a_1/b_1 \in R$ and $y = a_2/b_2 \in R$ with odd denominators b_1, b_2. R is closed under summation.

$$\begin{aligned}
&x + y = \frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1 b_2 + a_2 b_1}{b_1 b_2} \\
&\because b_1, b_2\ are\ odd \\
&\therefore b_1 b_2\ is\ odd \\
&\Rightarrow x + y \in R.
\end{aligned}$$

R is closed under multiplication.

$$\begin{aligned}
&xy = \frac{a_1}{b_1} \times \frac{a_2}{b_2} = \frac{a_1 a_2}{b_1 b_2} \\
&\because b_1, b_2\ are\ odd \\
&\therefore b_1 b_2\ is\ odd \\
&\Rightarrow xy \in R.
\end{aligned}$$

Based on bacis algebra, it is clear that $\langle R, +, -, 0\rangle$ forms an abelian group. The addition operation,$+$, is commutative, associative, and there is an inverse for any element in $\langle R, +, -, 0\rangle$. Also, $\langle R, \cdot, 1\rangle$ forms a monoid. The multiplication operation, $\times$, is associatvie and it distributes over addition. Also the multiplication is commutative. In summary, R forms a commutative ring.

Now let's consider the princeple ideal generated by 2, (2). Any $x \in R$ that can be written in the form of $x = 2a/b$, where b is odd, belongs in (2). Hence, $\forall x \in (2)$, x is a nonunit, since $b/2a \notin R$ by definition. In other words, we have:

$$(2) = \{x : \ x = \frac{a}{b}, \ where \ a \ is \ even, \ b \ is \ odd\}$$
$$R - (2) = \{x : \ x = \frac{a}{b}, \ where \ a, \ b \ are \ odd\}.$$

Let's assume there exists an element $y \in R-(2)$, such that y is a nonunit. Since $y \in R-(2)$, $y = a/b$, where a, b are both odd.

$$y \in R - (2) \Rightarrow y = \frac{a}{b}$$
$$\because a \ is \ odd$$
$$\therefore \frac{b}{a} \in R$$
$$\Rightarrow y \cdot \frac{b}{a} = \frac{a}{b} \cdot \frac{b}{a} = 1.$$

At this point, we've shown that R is a commutative ring with 1, for any $\forall x \in (2)$, x is a nonunit, and $\forall y \in R-(2)$, y is a unit. Now, we can apply the conclusion we just proved in the previous questin that if R is a commutative ring with 1 in which the set of all nonunits forms an ideal M, then R is a local ring with unique maximal ideal M. Here $M = (2)$ and R, the set of all rational numbers whose denominators is odd, forms a local ring.