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Math 611

In-Class Exercise (Lecture Aug 26th)

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In-Class Exercise

Question: Let $\langle L, \vee, \wedge, \rangle$ be a lattice. Define $x \leq y$ by $x \wedge y = x$.

a. Show that $\leq$ is a partial order.

Answer: According to the definition of partial order relation. I need to show that $\leq$ is reflexive, antisymmetric, and transitive. Since $\langle L, \vee, \wedge, \rangle$ is a lattice, the meet of an element with itself is itself. Hence $\leq$ is *reflexive*.

$$x \wedge x = x$$

$$\Rightarrow x \leq x\,.$$

Assume $x \leq y$ and $y \leq x$ both hold. We can prove that this assumption is followed by $x = y$. Hence the relation $\leq$ is *antisymmetric*.

$$x \leq y \Rightarrow x \wedge y = x$$

$$y \leq x \Rightarrow y \wedge x = x \wedge y = y$$

$$\Rightarrow x = y\,.$$

Assume $x \leq y$ and $y \leq z$ hold. We can derive that $x \leq z$ holds. The relation $\leq$ is *transitive*.

$$x \leq y \Rightarrow x \wedge y = x$$

$$y \leq z \Rightarrow y \wedge z = y$$

$$\begin{aligned} \Rightarrow x \wedge z &= (x \wedge y) \wedge z \\ &= x \wedge (y \wedge z) \\ &= x \wedge y \\ &= x \end{aligned}$$

$$\Rightarrow x \leq z\,.$$

In summary, if $\langle L, \vee, \wedge, \rangle$ is lattice, $x \leq y$ is defined by $x \wedge y = x$, then $\leq$ is a partial order relation.

b. Show that $glb(x, y) = x \wedge y$ and $lud(x, y) = x \vee y$.

Answer: We'll first show $glb(x, y) = x \wedge y$. Let $glb(x, y) = z$. Since we've shown $\le$ is a partial order relation, according to the definition of the greatest lower bound, we have:

$$z \le x, z \le y$$

$$\text{if } z' \le x \text{ and } z' \le y \text{ then } z' \le z\,.$$

Using the first condition, $z \le x$ and $z \le y$, we can derive $z \le x \wedge y$.

$$\begin{aligned}
& z \le x, z \le y \\
& \Rightarrow x \wedge z = z, y \wedge z = z \\
& \Rightarrow x \wedge z = x \wedge (y \wedge z) \\
& \qquad\qquad = (x \wedge y) \wedge z \\
& \qquad\qquad = z \\
& \Rightarrow z \le x \wedge y\,.
\end{aligned}$$

Using the second condition, if $z' \le x$ and $z' \le y$ then $z' \le z$, we can derive that $x \wedge y \le z$.

$$\begin{aligned}
& x \wedge (x \wedge y) = (x \wedge x) \wedge y \\
& \qquad\qquad\quad = x \wedge y \\
& \Rightarrow x \wedge y \le x \\
& y \wedge (x \wedge y) = y \wedge (y \wedge x) \\
& \qquad\qquad\quad = (y \wedge y) \wedge x \\
& \qquad\qquad\quad = y \wedge x \\
& \qquad\qquad\quad = x \wedge y \\
& \Rightarrow x \wedge y \le y \\
& \Rightarrow x \wedge y \le z\,.
\end{aligned}$$

At this point, we have $z \le x \wedge y$ and $x \wedge y \le z$. According to the *antisymmetric* rule of a partial order set, the two elements, z and $x \wedge y$ are the same. Hence $x \wedge y = z = glb(x, y)$.

Similarly we can show that $lup(x, y) = x \vee y$. But first we will prove that $a \wedge b = a$ if and only if $a \vee b = b$. We will prove with the *absorption* law on the lattice operations.

$$\begin{aligned}
& a \wedge b = a \\
& \Rightarrow (a \wedge b) \vee b = b = a \vee b\,.
\end{aligned}$$

In other words, according to its definition, the partial order relation, $\leq$, has the following relations with the lattice operations, $\wedge$ and $\vee$.

$$a \leq b \Leftrightarrow a \wedge b = a$$

$$a \leq b \Leftrightarrow a \vee b = b\,.$$

Let $lup(x, y) = z$. Since we've shown $\leq$ is a partial order relation, according to the definition of the least upper bound, we have:

$$x \leq z, y \leq z$$

$$\text{if } x \leq z' \text{ and } y \leq z' \text{ then } z \leq z'\,.$$

Using the first condition, $x \leq z$ and $y \leq z$, we can derive $x \vee y \leq z$.

$$x \leq z, y \leq z$$

$$\Rightarrow x \vee z = z, y \vee z = z$$

$$\Rightarrow x \vee z = x \vee (y \vee z)$$

$$= (x \vee y) \vee z$$

$$= z$$

$$\Rightarrow x \vee y \leq z\,.$$

Using the second condition, if $x \leq z'$ and $y \leq z'$ then $z \leq z'$, we can derive that $z \leq x \vee y$.

$$x \vee (x \vee y) = (x \vee x) \vee y$$

$$= x \vee y$$

$$\Rightarrow x \leq x \vee y$$

$$y \vee (x \vee y) = y \vee (y \vee x)$$

$$= (y \vee y) \vee x$$

$$= y \vee x$$

$$= x \vee y$$

$$\Rightarrow y \leq x \vee y$$

$$\Rightarrow z \leq x \vee y\,.$$

At this point, we have $x \vee y \leq z$ and $z \leq x \vee y$. According to the *antisymmetric* rule of a partial order set, the two elements, z and $x \vee y$ are the same. Hence $x \vee y = z = lub(x, y)$.