Student: Yu Cheng (Jade)

Math 611

Midterm Extra

November 20, 2010

Midterm Exam Question 3, 5, and 6

Question: Show that if $N \leq Z(G)$ and G/N is abelian, then G is nilpotent of class at most 2.

Answer: Let G' be the commutator of group G, $G' = [G, G]$. We know that commutator is the smallest normal subgroup of G such that the quotient group is abelian. In other words, if $H \triangleleft G$ and G/H is abelian, then $G' \leq H$.

In this problem, we have the following relation:

$$\begin{aligned} N \leq Z(G) \quad &\Rightarrow \quad N \triangleleft G \text{ and } G/N \text{ is abelian} \\ &\Rightarrow \quad G' \leq N \\ &\Rightarrow \quad G' \leq Z(G) \\ &\Rightarrow \quad [G', G] = \{e\}. \end{aligned}$$

Therefore we've shown that in the following series, G^2 is $\{e\}$.

$$G^0 = G,\ G^1 = [G, G],\ G^2 = [G^1, G], \cdots,\ G^{i+1} = [G^i, G].$$

If $G^1 = G' = \{e\}$, then group G is nilpotent of class 1. If $G^1 = G' \neq \{e\}$, then group G is nilpotent of class 2. In summary, group G is a nilpotent group of class at most 2.

Question: Let G be a group of order $255 = 3 \cdot 5 \cdot 17$ and let P be a Sylow 17 subgroup.

a. Prove that P is normal in G.

Answer: According to the Sylow Theorem, the number of Sylow p subgroups is congruent to 1 mod p.

$$n_{17} = 1, 18, 35 \cdots.$$

Also, if $|G| = p^k m$, then the number of Sylow p subgroups divides m.

$$n_{17} \mid 15 \Rightarrow n_{17} = 1.$$

Since the only choice for $|Syl_{17}(G)|$ is 1, we've shown $P \triangleleft G$.

b. Using the fact that $|Aut(P)| = 16$, prove that $P \leq Z(G)$.

Answer: Let's define a map φ_g as $\varphi_g(p) = gpg^{-1}$, where $g \in G$ and $p \in P$. Since P is normal, $gpg^{-1} \in P$ for $\forall g \in G$, φ_g is, therefore, a map from P to P. It is also easy to show that φ_g is a group homomorphism. In other words, $\varphi_g \in Aut(P)$

$$\begin{aligned}
&\varphi_g\left(p_1p_2\right) = gp_1p_2g^{-1} \\
&\varphi_g\left(p_1\right)\varphi_g\left(p_2\right) = gp_1g^{-1}gp_2g^{-1} = gp_1p_2g^{-1} \\
&\Rightarrow \varphi_g\left(p_1p_2\right) = \varphi_g\left(p_1\right)\varphi_g\left(p_2\right).
\end{aligned}$$

Now let's assume $g^k = e$, where $g \in G$, $|g| = k$. Since k is in fact the order of the cyclic subgroup of G generated by g, according to the Lagrange's Theorem, $k \mid G$. Let's also assume $|\varphi_g| = 2^r$. Since the order of φ_g has to divide $|Aut(P)| = 2^4$, we have $r \in \{0, 1, 2, 3, 4\}$. We can show that $|\varphi_g|$ divides $|g|$.

$$\begin{aligned}
\left(\varphi_g\right)^k(x) &= \left(\varphi_g\right)^{k-1} \circ \varphi_g(x) \\
&= \left(\varphi_g\right)^{k-1}\left(gxg^{-1}\right) \\
&= \left(\varphi_g\right)^{k-2} \circ \varphi_g\left(gxg^{-1}\right) \\
&= \left(\varphi_g\right)^{k-2}\left(ggxg^{-1}g^{-1}\right) \\
&= \cdots = \underbrace{g\cdots g}_{k}\, x \underbrace{g^{-1}\cdots g^{-1}}_{k} \\
&= x.
\end{aligned}$$

Hence, we've shown that $\left(\varphi_g\right)^k$ is the identify map, $\left(\varphi_g\right)^k = e_{Aut(G)}$. At the same time $\left(\varphi_g\right)^{2r} = e_{Aut(G)}$, and $2r$ is the smallest such power by definition. So we have $2r \mid k$. Meanwhile, $k \mid |G|$.

$$\begin{aligned}
&2r \mid k,\ and\ k \mid |G| \\
&\Rightarrow 2r \mid |G| \\
&\Rightarrow 2r \mid (3 \cdot 5 \cdot 17).
\end{aligned}$$

Prime factor 2 is not in $3 \cdot 5 \cdot 17$, hence $r = 0$. So, $|\varphi_g| = 2^0 = 1$, in other words, $\varphi_g = e_{Aut(G)}$.

$$\begin{aligned}
&\varphi_g(p) = p \\
&\Rightarrow gpg^{-1} = p
\end{aligned}$$

$$\Rightarrow gp = pg,\ \forall g \in G,\ and\ \forall p \in P$$
$$\Rightarrow P \leq Z(G).$$

c. Show that G is nilpotent.

Answer: Since $P \leq Z(G)$, $|P| = 17$, and $|G| = 3 \cdot 5 \cdot 17$, It is clear that there are three possible orders of the center $Z(G)$:

Case-1 $|Z(G)| = 17 \cdot 3$.

$$|Z(G)| = 17 \cdot 3$$
$$\Rightarrow |G/Z(G)| = 5$$
$$\Rightarrow G/Z(G)\ is\ abelian$$
$$\Rightarrow G/Z(G)\ is\ nilpotent$$
$$\Rightarrow G\ is\ nilpotent.$$

Case-2 $|Z(G)| = 17 \cdot 5$.

$$|Z(G)| = 17 \cdot 5$$
$$\Rightarrow |G/Z(G)| = 3$$
$$\Rightarrow G/Z(G)\ is\ abelian$$
$$\Rightarrow G/Z(G)\ is\ nilpotent$$
$$\Rightarrow G\ is\ nilpotent.$$

Case-3 $|Z(G)| = 17 \cdot 3 \cdot 5$.

$$|Z(G)| = 17 \cdot 3 \cdot 5$$
$$\Rightarrow |Z(G)| = |G|$$
$$\Rightarrow Z(G) = G$$
$$\Rightarrow G\ is\ abelian$$
$$\Rightarrow G\ is\ nilpotent.$$

In summary, we've shown that in any of the three possible cases, G is a nilpotent group.

d. Show that G is cyclic.

Answer: A finite nilpotent group G is the direct product of its Sylow subgroups. The Slow subgroups of G all have prime orders, 17, 3, and 5. A group of prime order is isomorphic to the quotient of the group of integers. So, we have Sylow 17 subgroup P_{17}, Sylow 3 subgroup, P_3, and Sylow 5 subgroup, P_5 satisfying the following relationships:

$$P_{17} \cong \mathbb{Z}_{17}$$
$$P_3 \cong \mathbb{Z}_3$$
$$P_5 \cong \mathbb{Z}_5.$$

Therefore, $G = P_{17} \times P_3 \times P_5 = \mathbb{Z}_{17} \times \mathbb{Z}_3 \times \mathbb{Z}_5$. At the same time, since $GCD(3, 5, 17) = 1$, $\mathbb{Z}_{17} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{17\cdot 3\cdot 5} = \mathbb{Z}_{225}$, which is cyclic.

$$G = P_{17} \times P_3 \times P_5 \cong \mathbb{Z}_{225}$$
$$\Rightarrow G \ is \ cyclic.$$

Question: Show that there is no simple group of order p^3q, where p and q are distinct primes.

Answer: According to the Sylow Theorem, the number of Sylow p subgroups is congruent to 1 mod p.

$$n_p = 1, p+1, 2p+1 \cdots$$
$$n_q = 1, q+1, 2q+1 \cdots.$$

Also, if $|G| = p^k m$, then the number of Sylow p subgroups divides m. So we have the following conditions, where, $k_1, k_2 \in 0, 1, 2, \cdots$

$$n_p \mid q \Rightarrow (k_1 p + 1) \mid q$$
$$n_q \mid p^3 \Rightarrow (k_2 q + 1) \mid p^3.$$

Case-1 $q < p$. Since $(k_1 p + 1) \mid q$, we have the following inequality:

$$\begin{aligned} (k_1 p + 1) \mid q &\Rightarrow k_1 p + 1 \le q \\ &\Rightarrow k_1 p + 1 < p \\ &\Rightarrow k_1 p < p - 1 \\ &\Rightarrow (1 - k_1)\, p > 1 \\ &\Rightarrow k_1 = 0 \\ n_p &= k_1 p + 1 = 1. \end{aligned}$$

So $|Syl_p(G)| = 1$, therefore, $P \triangleleft G$, where $P \in Syl_p(G)$, $|P| = p^3$. Hence we've found a non-trivial normal subgroup of G, G is not simple.

Case-2 $q > p$. We have $n_q \mid p^3$. There are only four possible n_q's, that divide p^3. They are discussed below one by one.

Case-2-1 $n_q = 1$. If this is the case, then we're done, since we've found a non-trivial normal subgroup of $|G|$. Namely, $Q \triangleleft G$, where $Q \in Syl_q(G)$, $|Q| = q$.

Case-2-2 $n_q = p$. Since $n_q = k_2 q + 1$, we have:

$$\begin{aligned} n_q &= k_2 q + 1 = p \\ &\Rightarrow k_2 q + 1 < q \\ &\Rightarrow (1 - k_2)\, q < 1 \\ &\Rightarrow k_2 = 0 \\ n_q &= k_2 q + 1 = 1 \end{aligned}$$

As shown in Case-2-1, G is not simple since it has a non-trivial normal subgroup Q, where Q is the only element in $Syl_q(G)$.

Case-2-3 $n_q = p^2$. Since $n_q = k_2 q + 1$, we have:

$$\begin{aligned} &n_q = k_2 q + 1 = p^2 \\ &\Rightarrow k_2 q = p^2 - 1 = (p - 1)\,(p + 1)\,. \end{aligned}$$

So the prime factor q is in $(p-1)(p+1)$. Since $p - 1 < q - 1 < q$, q must be in $p+1$. In other words, $q \mid (p+1)$ with p, q are both prime numbers. The only combination of prime numbers that satisfy this condition is $p = 2$ and $q = 3$.

The problem is converted to proving there is no simple group order $|G| = 2^3 \cdot 3 = 24$. According to the Sylow Theorem, the number of Sylow p subgroups is congruent to 1 mod p.

$$n_2 = 1, 3, 5 \cdots.$$

Also, if $|G| = p^k m$, then the number of Sylow p subgroups divides m.

$$n_2 \mid 3 \Rightarrow n_2 = 1 \textit{ or } 3.$$

If $n_2 = 3$ then we can define a group action of G acting on the coset space of the Sylow 2 subgroup, P. $[G : P] = 3$. With this, we see that $|G| \le S_3$, and derive the following contradiction:

$$|G| = 24 \le |S_3| = 6.$$

Hence $n_2 \neq 3$, so $n_2 = 1$. As discussed previously, we've found a non-trivial normal subgroup of $|G|$. Namely, the Slow 2 subgroup. So $|G|$ is not simple.

Case-2-4 $n_q = p^3$. We count the number of elements with order q. It is $(q-1) \cdot p^3$. Hence the number elements left is:

$$\begin{aligned} |G| - (q-1)\cdot p^3 &= q\cdot p^3 - (q-1)\cdot p^3 \\ &= p^3. \end{aligned}$$

An element count of p^3 is enough for only one Sylow p subgroup. Therefore in this case, we derived that $n_p = 1$. As shown in Case 1, G is not simple since it has a non-trivial normal subgroup P, where $P \in Syl_p(G)$.

In summary, we've discussed every possible p's and q's, and the conclusion is that there is no simple group of order p^3q.