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ICS 241

Recitation Lecture Note #7

October 8, 2009

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Question: Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack. [Chapter 8.5 Review]

a. $\{(f, g) \mid f(1) = g(1)\}$

Answer: Equivalence relation. We need to observe whether the relation is relation reflexive, $f(1) = f(1)$ holds, symmetric , $f(1) = g(1) \Rightarrow g(1) = f(1)$, and transitive $f(1) = f'(1), f'(1) = f''(1) \Rightarrow f(1) = f''(1)$. Therefore all three properties hold.

b. $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$

Answer: Not an equivalence relation because it is not transitive. For instance, $f(0) = f'(0)$ and $f'(1) = f''(1) \nRightarrow f(0) = f''(0)$ or $f(1) = f''(1)$

c. $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$

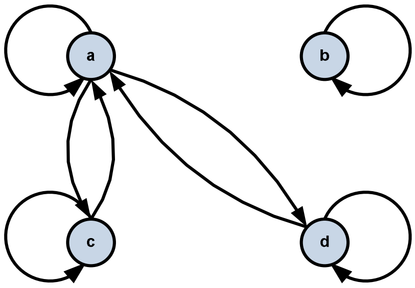
Answer: Not an equivalence relation because it is neither reflexive, symmetric nor transitive.

d. $\{(f, g) \mid \text{for some } C \in \mathbb{Z}, \text{for all } x \in \mathbb{Z},\ f(x) - g(x) = C\}$

Answer: Equivalence relation. We need to observe whether the relation is relation reflexive, $f(x) - f(x) = 0\ (C = 0)$, symmetric, $f(x) - g(x) = C \Rightarrow g(x) - f(x) = -C$, and transitive $f(x) - f'(x) = C_1$ and $f'(x) - f''(x) = C_2 \Rightarrow f(x) - f''(x) = C_1 + C_2$. Therefore all three properties hold.

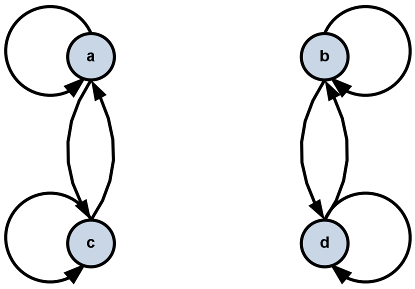
Question: Determine whether the relation with the directed graphs shown in an equivalence relation. [Chapter 8.5 Review]

a.



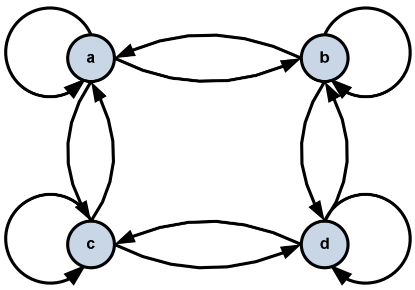
Answer: No, this directed graph does not represent an equivalence relation because it is not transitive. For instance, vertex c is related with vertex a, vertex a is related with vertex d, but vertex c is not shown to be related with vertex d.

b.



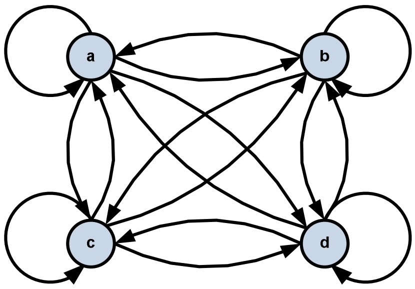
Answer: Yes, this graph represents an equivalence relation. We need to observe whether the relation is relation reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate – an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1). We see that this relation is an equivalence relation, satisfying all three properties. The equivalence classes are $\{a, d\}$ and $\{b, c\}$.

c.



Answer: No, this graph does not represent an equivalence relation. It is not transitive. For instance, vertex a is related with vertex c, vertex c is related with vertex d, but vertex a is not shown to be related with vertex d.

d.

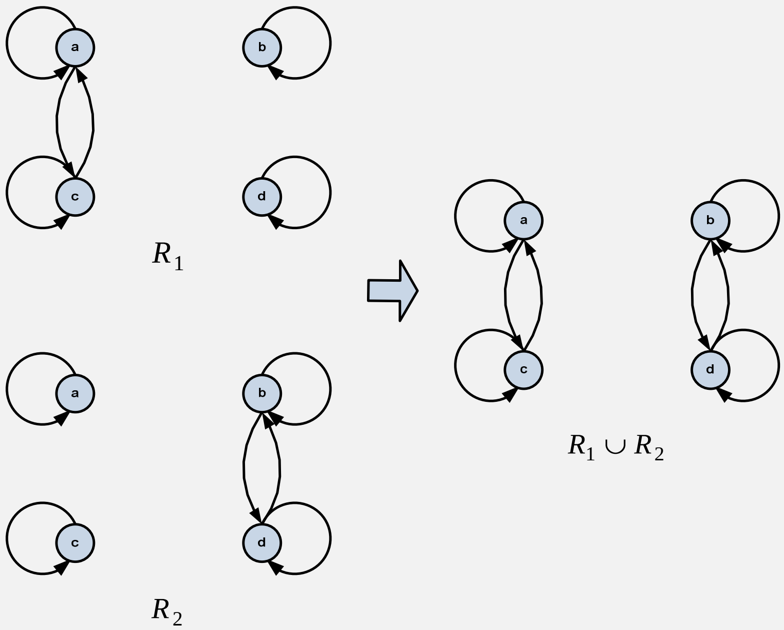


Answer: Yes, this graph represents a equivalence relation. We need to observe whether the relation is relation reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate – an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1). We see that this relation is an equivalence relation, satisfying all three properties. There is only one equivalence class, which contains all four vertices $\{a, b, c, d\}$.

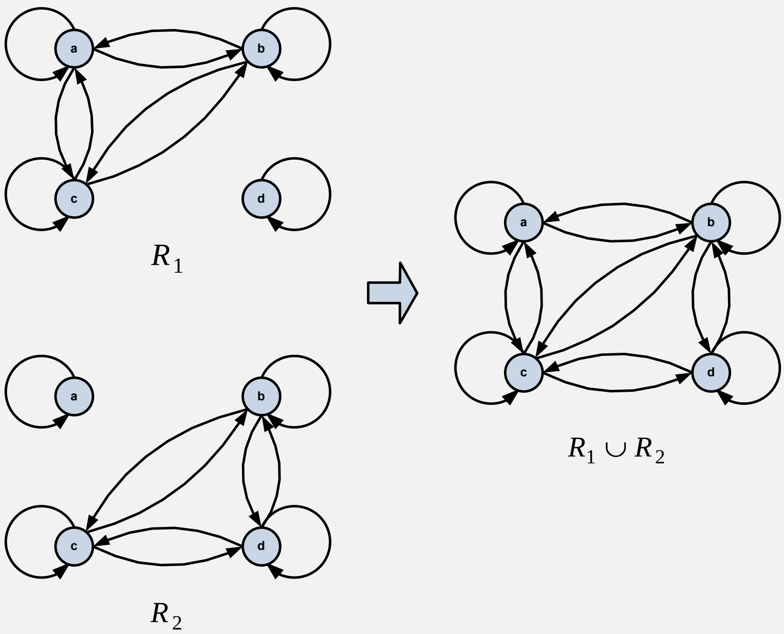
Question: Suppose that R_1 and R_2 are equivalence relations on the set S. Determine whether each of these combination of R_1 and R_2 must be an equivalence relation. [Chapter 8.5 Review]

a. $R_1 \cup R_2$

Answer: This is not necessarily be a equivalence relation, since it is not necessarily be transitive. Let look at two examples. In the first example $\boldsymbol{R_1} \cup \boldsymbol{R_2}$ is an equivalence relation, while in the second example it is not. Therefore set R_1 and R_2 are both equivalence relations is not a necessary and sufficient condition for $\boldsymbol{R_1} \cup \boldsymbol{R_2}$ being a equivalence relation.



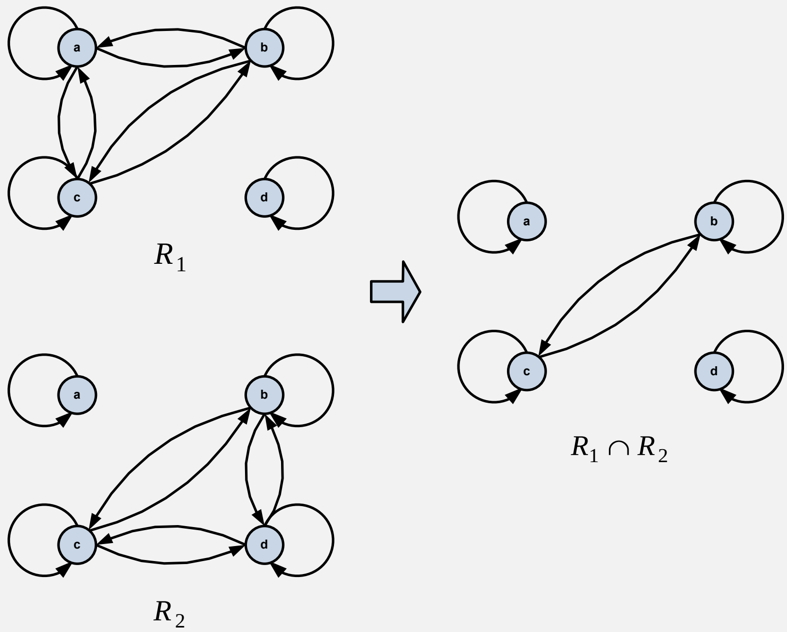
The diagram above shows that $R_1 \cup R_2$ is also an equivalence relation for these two input graphs.



The diagram above shows that $R_1 \cup R_2$ is not an equivalence relation for these two input graphs.

b. $R_1 \cap R_2$

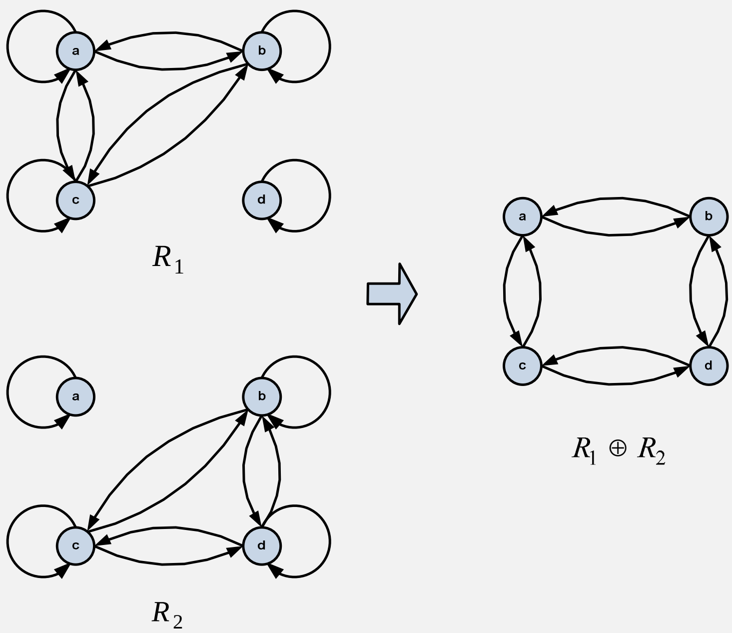
Answer: Since the intersection of reflexive, symmetric, and transitive relations also has these properties, the intersection of equivalence relations is an equivalence relation. Let's look at the following example.



In other words, set R_1 and R_2 are both equivalence relations is a necessary and sufficient condition for $R_1 \cap R_2$ being an equivalence relation

c. $R_1 \oplus R_2$

Answer: This will never be an equivalence relation on a nonempty set, since it is not reflexive.



In other words, set R_1 and R_2 are both equivalence relations is a necessary and sufficient condition for $R_1 \oplus R_2$ being a non-equivalence relation.

Question:	Let $A = \{(x, y) \mid x, y \text{ integers}\}$. Define a relation R on A by the rule [Chapter 8.6 Review]

$$(a, b)R(c, d) \leftrightarrow a \leq c \text{ or } b \leq d$$

Determine whether R is a partial order relation on A.

Answer: R is reflexive: $(a, b)R(a, b)$ for all elements (a, b) because $a \leq a$ and $b \leq b$ is always true. R is not antisymmetric: For example, $(1,4)R(3,2)$ because $1 \leq 3$, and $(3,2)R(1,4)$ because $2 \leq 4$. But $(1,4) \neq (3,2)$. R is not transitive: For example, $(1,4)R(3,2)$ because $1 \leq 3$, and $(3,2)R(0,3)$ because $2 \leq 3$. But $\left((1,4), (0,3)\right) \notin R$ because $1 \nleq 0$ and$4 \nleq 3$. Therefore, R is not a partial order relation because R is neither antisymmetric nor transitive.

Question: Let $A = \{(x, y) \mid x, y \text{ integers}\}$. Define a relation R on A by the rule [Chapter 8.6 Review]

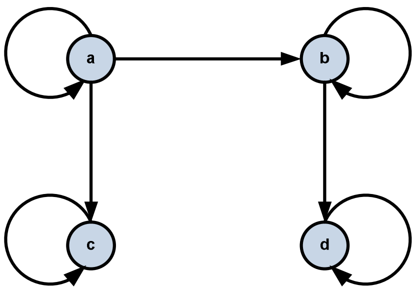
$$(a, b)R(c, d) \leftrightarrow a = c \text{ or } b = d$$

Determine whether R is a partial order relation on A.

Answer: R is reflexive: $(a, b)R(a, b)$ for all elements (a, b) because $a = a$ and $b = b$ is always true. R is not antisymmetric: For example, $(1,2)R(1,3)$ because $1 = 1$, and $(1,3)R(1,2)$ because $1 = 1$. R is not transitive: For example, $(1,2)R(1,3)$ because $1 = 1$, and $(1,3)R(4,3)$ because $3 = 3$. But $\left((1,2), (4,3)\right) \notin R$ because $1 \neq 4$ and$4 \neq 3$. Therefore, R is not a partial order relation because R is neither antisymmetric nor transitive.

Question: Determine whether the relation with the directed graph shown is a partial order. [Chapter 8.6 Review]

a.

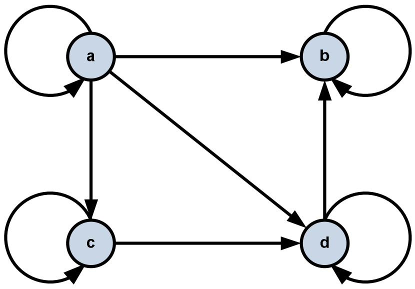


Answer: No, this directed graph does not represent a partial order. We need to observe whether the relation is relation reflexive (there is a loop at each vertex), antisymmetric (every edge that appears is not accompanied by its antiparallel mate – an edge involving the same two vertices but pointing in the opposite direction), and transitive, which is not satisfied. For instance, vertex a is related with vertex b, vertex b is related with vertex d, but vertex a is not shown to be related with vertex d.

We can also investigate the problem in the perspective of the zero-one matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The matrix representation shows that it satisfies all three properties.

b.

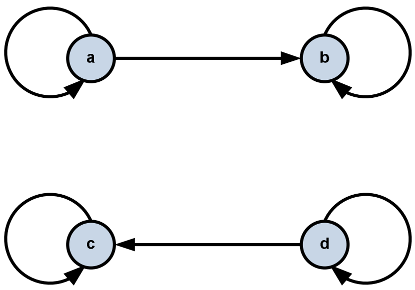


Answer: No, this graph does not represent a partial order. The transitive property is not satisfied. For instance, vertex c is related with vertex d, vertex d is related with vertex b, but vertex a is not shown to be related with vertex b.

We can also investigate the problem in the perspective of the zero-one matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

The matrix representation shows that it satisfies all three properties.

c.



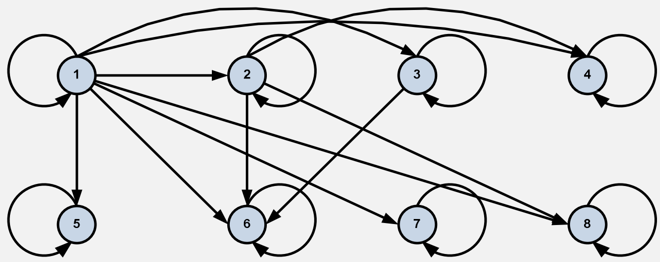
Answer: Yes, this graph represents a partial order. It satisfies all three properties: reflexive, antisymmetric, and transitive.

We can also investigate the problem in the perspective of the zero-one matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

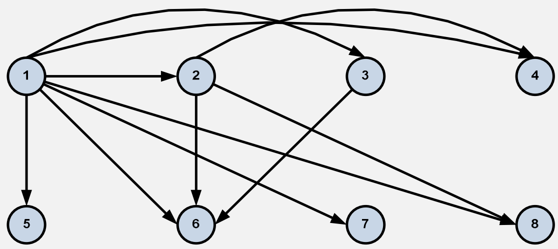
The matrix representation shows that it satisfies all three properties.

Question: Draw the Hasse diagram for divisibility on the set {1, 2, 3, 4, 5, 6, 7, 8}. [Chapter 8.6 Review]

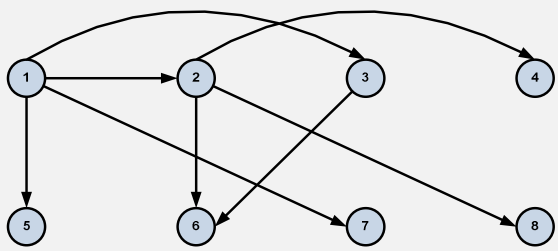
Answer: First list out the relations: {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (4,8), (5,5), (6,6), (7,7), (8,8)}. Then draw the directed graph representing this relation:



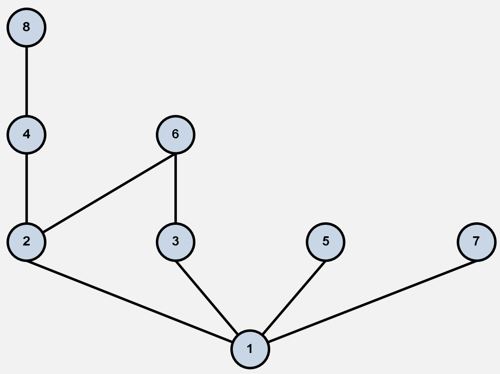
Then we would delete all loops around the vertices because we know the graph is reflexive. Therefore we don't have to show the edges representing this property.



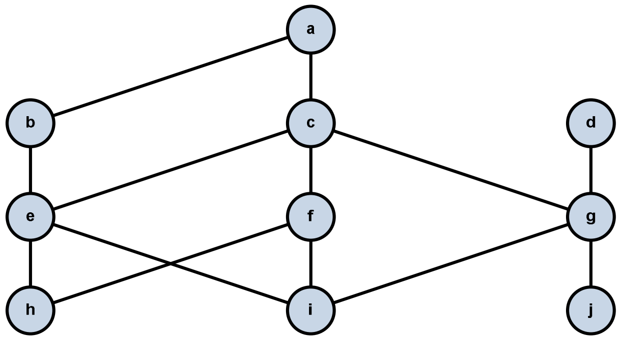
After this, we can delete the edges that must be present because of the transitive property.



Finally, we don't have to show the direction of the edges if the edges are all pointed "upward":



Question: Draw Referring to this Hasse diagram of a partially ordered set, find the following: [Chapter 8.6 Review]



a. Find the maximum elements.

Answer: a and d. An element of a poset is called maximal if it is not less than any element of the poset. In other words, the maximal elements are the ones with no other elements above them.

b. Find the minimal elements.

Answer: $h, i,$ and j. An element of a poset is call minimal if it is not greater than any element of the poset. In other words, the minimal elements are the ones with no other elements below them.

c. Is there a greatest element?

Answer: No, since neither a nor b is greater than the other. An element of a poset is called the greatest element if it is greater than every other element.

d. Is there a least element?

Answer: No, since neither h, i nor j is less than the others. An element of a poset is called the least element if it is less than every other element.

e. Find all upper bounds of $\{d, e\}$.

Answer: No such element. An element is called the upper bound of a set A if it is greater than or equals to all elements in the set A. In other words, we need to find elements from which we can find downward paths to d and e.

f. Find the least upper bounds of $\{d, e\}$.

Answer: No such element since there's no upper bounds. An element is called the least upper bound (lub) of the subset A if it is an upper bound that is less than every other upper bound of A.

g. Find all lower bounds of $\{a, e, g\}$.

Answer: i is the only lower bound. An element is called the lower bound of a set A if it is less than or equals to all elements in the set A. In other words, we need to find elements from which we can find upward paths to a, e and g.

h. Find the greatest lower bounds of $\{a, e, g\}$.

Answer: i since it's the only lower bound. An element is called the greatest lower bound (glb) of the subset A if it is an lower bound that is greater than every other lower bound of A.

i. Find the greatest lower bounds of $\{b, c, f\}$.

Answer: Both h and i are lower bounds of $\{b, c, f\}$. But there is no greatest lower bound

j. Find the least upper bounds of $\{h, i, j\}$.

Answer: Both a and c are upper bounds of $\{h, i, j\}$. The element c is the least upper bound.

k. Find the greatest lower bound of $\{g, h\}$.

Answer: There is no lower bound of $\{g, h\}$. Hence there is no greatest lower bound.

l. Find the least upper bound of $\{f, i\}$

Answer: The elements a, c, and f are upper bounds of $\{f, i\}$. The element f is the least upper bound.