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Math 311

Homework #1

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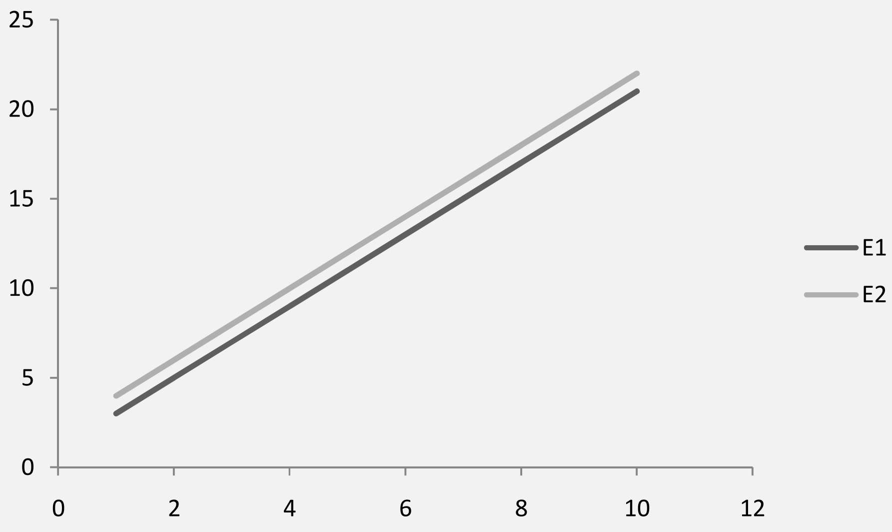
Chapter 1.1

Question 2, 6: Are the following equations linear?

Answer: $x_1 \cdot x_2 + x_2 = 1$ is not a linear equation. $\pi \cdot x_1 + \sqrt{7}x_2 = \sqrt{3}$ is a linear equation. Because the first one doesn't satisfy the general form of linear equation, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, while the second one does.

Question 12: Sketch a graph for each equation to determine whether the system has a unique solution, no solution, or infinitely many solutions.

Answer: $2x - y = -1, 2x - y = 2$



There is no solution for this system, since the equations do not intersect. There is no point on the graph that satisfy both equations at the same time.

Question 15: The (2×3) system of linear equations:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

is represented geometrically by two planes. How are the planes related when:

a. The system has no solution?

Answer: When the planes are parallel and do not intersect. Then there would be no point in the space that satisfies both equations, there is, therefore, no solution for the system. For example, the following system has no solution:

$$2x + 3y + 2x = 3$$

$$4x + 6y + 4z = 1$$

b. The system has infinitely many solutions?

Answer: When the planes are not parallel and intersect. Then all points on the intersected line satisfy both equations, there are, therefore, infinitely many solutions for the system. For example, the following system has infinitely many solutions:

$$x + 2y - x = 2$$

$$x + y + z = 3$$

c. Is it possible for the system to have a unique solution? Explain.

Answer: It is no possible for the system to have a unique solution, since two planes cannot intersect in a point, which is the case when there is a unique solution.

Question 30: Display the augmented matrix for the given system. Use elementary operations on equations to obtain an equivalent system of equations in which x_1 appears in the first equation with coefficient one and has been eliminated from the remaining equations. Simultaneously, perform the corresponding elementary row operations on the augmented matrix

$$2x_1 + 3x_2 = 6$$

$$4x_1 - x_2 = 7$$

Answer: Starting from the augmented matrix:

$$\begin{bmatrix} 2 & 3 & 6 \\ 4 & -1 & 7 \end{bmatrix} \rightarrow (1/2R_1) \rightarrow \begin{bmatrix} 1 & 1.5 & 3 \\ 4 & -1 & 7 \end{bmatrix} \rightarrow (R_2 - 4R_1) \rightarrow \begin{bmatrix} 1 & 1.5 & 3 \\ 0 & -7 & -5 \end{bmatrix}.$$

Question 38: Consider the (2×2) system

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 - a_{22}x_2 = b_2$$

Show that if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then this system is equivalent to a system of the form

$$c_{11}x_1 + c_{12}x_2 = d_1$$

$$c_{22}x_2 = d_2$$

Where $c_{11} \neq 0$ and $c_{22} \neq 0$. Note that the second system always has a solution.

Answer: <u>*Case 1:*</u> When $c_{11} = 0$.

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad a_{21}x_1 - a_{22}x_2 = b_2$$

$$a_{21}x_1 - a_{22}x_2 = b_2 \qquad a_{12}x_2 = b_1$$

To prove $a_{11}a_{22} - a_{12}a_{21} \neq 0$ ensures the transformation, we need to prove $a_{12}a_{21} \neq 0$, which is the same as proving $a_{12} \neq 0$ and $a_{21} \neq 0$ ensures the transformation. Comparing the second from with the target form, we have $a_{21} = c_{11}$, $a_{22} = c_{12}$, and $a_{12} = c_{22}$. Then $a_{12} = c_{22} \neq 0$, and $a_{21} = c_{11} \neq 0$. They are the condition for the second system. We, therefore proved that $a_{11}a_{22} - a_{12}a_{21} \neq 0$ ensures the transformation.

<u>*Case 2:*</u> When $c_{11} \neq 0$

$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - \dfrac{a_{21}a_{12}}{a_{11}} & b_2 - \dfrac{a_{21}b_1}{a_{11}} \end{bmatrix}$$

Comparing the augmented matrix above with $\begin{bmatrix} c_{11} & c_{12} & d_1 \\ 0 & c_{22} & d_2 \end{bmatrix}$, we have $c_{11} = a_{11}$, and $c_{22} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$. In order to satisfy $c_{11} \neq 0$ and $c_{22} \neq 0$, we just need to satisfy:

$$a_{22} - \frac{a_{21}a_{12}}{a_{11}} \neq 0$$

$$a_{22}a_{11} - a_{21}a_{12} \neq 0$$

Therefore we proved the second case. $a_{11}a_{22} - a_{12}a_{21} \neq 0$ ensures the transformation.

Question 4-10: Consider the following matrices,

a. Either state that the matrix is in echelon form or use elementary row operations to transform it to echelon form.

b. If the matrix is in echelon form, transform it to reduced echelon form.

Answer: $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ is not in echelon form because the left bottom corner entry is not 0 while the entry on top of it is. Convert it to echelon form:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \to (R_1 \leftrightarrow R_2) \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

$\begin{bmatrix} 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ is not in echelon form because it's not a leading one in the first row. Convert it to echelon form:

$$\begin{bmatrix} 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \left(\frac{1}{2}R_1\right) \to \begin{bmatrix} 1 & 0 & 1.5 & 0.5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ is not in echelon form because it's not a leading one in row one and row three. Convert it to echelon form:

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \to \left(\frac{1}{2}R_1\right) \to \begin{bmatrix} 1 & -0.5 & 1.5 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \to \left(-\frac{1}{3}R_3\right) \to \begin{bmatrix} 1 & -0.5 & 1.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\begin{bmatrix} -1 & 4 & -3 & 4 & 6 \\ 0 & 2 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ is not in echelon form because it's not a leading one in row one and row two. Convert it to echelon form:

$$\begin{bmatrix} -1 & 4 & -3 & 4 & 6 \\ 0 & 2 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to (-R_1) \to \begin{bmatrix} 1 & -4 & 3 & -4 & -6 \\ 0 & 2 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \left(\frac{1}{2}R_2\right) \to \begin{bmatrix} 1 & -4 & 3 & -4 & -6 \\ 0 & 1 & 0.5 & -1.5 & -1.5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Question: Solve the system by transforming the augmented matrix to reduced echelon form.

24. $x_1 - x_2 + x_3 = 3$
$2x_1 + x_2 - 4x_3 = -3$

Answer: $\begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 1 & -4 & -3 \end{bmatrix} \to (\mathrm{R}_2 - 2R_1) \to \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 3 & -6 & -9 \end{bmatrix}$

$$\to \left(\frac{1}{3}R_2\right) \to \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -3 \end{bmatrix}$$

$$\to (\mathrm{R}_1 + R_2) \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -3 \end{bmatrix}.$$

Therefore, $x_1 = x_3$; $x_2 = 2x_3$, and x_3 is arbitrary.

26. $x_1 - x_2 + x_3 = 4$
$2x_1 - 2x_2 + 3x_3 = 2$

Answer: $\begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & -2 & 3 & 2 \end{bmatrix} \to (\mathrm{R}_2 - 2R_1) \to \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix}$

$$\to (\mathrm{R}_1 - R_2) \to \begin{bmatrix} 1 & -1 & 0 & 10 \\ 0 & 0 & 1 & -6 \end{bmatrix}.$$

Therefore, $x_1 = x_2 + 10$; $x_3 = -6$, and x_2 is arbitrary.

Question 56: Find the formula for the sum.

Answer: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n \cdot (n+1) \cdot (2n+1)$.

Chapter 1.3

Question 2,4: Transform the augmented matrix for the given system to reduced echelon form and, in the notation of Theorem 3, determine n, r, and the number, $n - r$, of independent variables. If $n - r > 0$, then identify $n - r$ independent variables.

a. $2x_1 + 2x_2 = 1$
$4x_1 + 5x_2 = 4$
$4x_1 + 2x_2 = -2$

Answer:

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 5 & 4 \\ 4 & 2 & -2 \end{bmatrix} \rightarrow (1/2\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 1 & 0.5 \\ 4 & 5 & 4 \\ 4 & 2 & -2 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_2 - 4\mathrm{R}_1) \rightarrow \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 2 \\ 4 & 2 & -2 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_3 - 4\mathrm{R}_1) \rightarrow \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_1 - \mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 0 & -1.5 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_3 + 2\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 0 & -1.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $n = 2, r = 2$, and $n - r = 0$. There's no independent variable in this system.

b.

$x_1 + 2x_2 + 3x_3 + 2x_4 = 1$

$x_1 + 2x_2 + 3x_3 + 5x_4 = 2$

$2x_1 + 4x_2 + 6x_3 + x_4 = 1$

$-x_1 - 2x_2 - 3x_3 + 7x_4 = 2$

Answer:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 5 & 2 \\ 2 & 4 & 6 & 1 & 1 \\ -1 & -2 & -3 & 7 & 2 \end{bmatrix} \rightarrow (\mathrm{R}_2 - \mathrm{R}_1) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 2 & 4 & 6 & 1 & 1 \\ -1 & -2 & -3 & 7 & 2 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_3 - 2\mathrm{R}_1) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 & -1 \\ -1 & -2 & -3 & 7 & 2 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_4 + \mathrm{R}_1) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 9 & 3 \end{bmatrix}$$

$$\rightarrow \left(\frac{1}{3}\mathrm{R}_2\right) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 9 & 3 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_1 - 2\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 9 & 3 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_3 + 3\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 3 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_4 - 9\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $n = 4, r = 2,$ and $n - r = 2$. There are two independent variables in this system. They are x_2 and x_3.

Question 8: A homogeneous system of 4 equations in 5 unknowns

Answer: $\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{25} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{25} & 0 \end{bmatrix}$, $n = 5$, $m = 4$, r can be one of the following numbers: $0, 1, 2, 3, 4$. And therefore $n - r$ can be one of the following numbers: $5, 4, 3, 2, 1$. So independent variable will always exist in this system. We have infinite solutions.

Question 10: A system of 4 equations in 3 unknowns

Answer: $\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & b_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & b_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & b_3 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & b_4 \end{bmatrix}$, $n = 3$, $m = 4$, r can be one of the following numbers: $0, 1, 2, 3, 4$. And therefore $n - r$ can be one of the following numbers: $3, 2, 1, 0$ and -1. When $n - r$ is $3, 2$ or 1, there are infinite solutions. When $n - r = 0$, there is a unique solution. When any row's in the form of: $[0, \cdots, 0, 1]$, the system is inconsistent and has no solution.

Question 12: A homogeneous system of 4 equations in 3 unknowns

Answer: $\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 0 \end{bmatrix}$, $n = 3$, $m = 4$, r can be one of the following numbers: $0, 1, 2, 3$. And therefore $n - r$ can be one of the following numbers: $3, 2, 1, 0$. When $n - r$ is $3, 2$ or 1, therefore infinite solutions. When $n - r = 0$, there is a unique solution, which is the trivial solution.

Question 14: A system of 3 equations in 4 unknowns, that has $x_1 = -1, x_2 = 0, x_3 = 2, x_4 = -3$ as a solution.

Answer: $\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & b_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & b_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & b_3 \end{bmatrix}$, $n = 4,\ m = 3$, r can be one of the following numbers: $0, 1, 2, 3$. And therefore $n - r$ can be one of the following numbers: $4, 3, 2, 1$. Since we already know that there is a solution, rows in the form of: $[0, \cdots, 0, 1]$, therefore, does not exist. So, this system has infinitely many solutions.

Question 24: Consider the system of equation:

$x_1 + 3x_2 - x_3 = \mathrm{b}_1$

$x_1 + 2x_2 = \mathrm{b}_2$

$3x_1 + 7x_2 - x_3 = \mathrm{b}_3$

a. Determine conditions on $b_1, b_2,$ and b_3 that are necessary and sufficient for the system to be consistent.

Answer:

$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ 1 & 2 & 0 & b_2 \\ 3 & 7 & -1 & b_3 \end{bmatrix} \rightarrow (\mathrm{R}_2 - \mathrm{R}_1) \rightarrow \begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 3 & 7 & -1 & b_3 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_3 - 3\mathrm{R}_1) \rightarrow \begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & -2 & 2 & b_3 - 3b_1 \end{bmatrix}$$

$$\rightarrow (-\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & 1 & -1 & b_1 - b_2 \\ 0 & -2 & 2 & b_3 - 3b_1 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_1 - 3\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 0 & -2 & 3b_2 - 2b_1 \\ 0 & 1 & -1 & b_1 - b_2 \\ 0 & -2 & 2 & b_3 - 3b_1 \end{bmatrix}$$

$$\rightarrow (\mathrm{R}_3 + 2\mathrm{R}_2) \rightarrow \begin{bmatrix} 1 & 0 & -2 & 3b_2 - 2b_1 \\ 0 & 1 & -1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - 2b_2 \end{bmatrix}.$$

According to Remark 1, we can't have $[0, \cdots, 0, 1]$ for a consistent system. So $b_3 - b_1 - 2b_2 = 0$. In this case we have:

$$\begin{bmatrix} 1 & 0 & -2 & 3b_2 - 2b_1 \\ 0 & 1 & -1 & b_1 - b_2 \end{bmatrix}$$

In this system, we have $n = 3, m = 2,$ and $r = 2$. We have one free variable x_3. System has infinitely many solutions.

b. In each of the following, either use your answer from **a** to show the system is inconsistent or exhibit a solution.

i. $b_1 = 1, b_2 = 1, b_3 = 3$

Answer: $b_3 - b_1 - 2b_2 = 3 - 1 - 2 = 0$. The reduced echelon form shown in **a** looks like:

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$x_1 = 2x_3 + 1$, $x_2 = x_3$, and x_3 is the free variable, which can be any number. A particular solution could be: $x_3 = 1, x_1 = 3, x_2 = 1$.

ii. $b_1 = 1, b_2 = 0, b_3 = -1$

Answer: $b_3 - b_1 - 2b_2 = -1 - 1 - 0 = -2 \neq 0$. The reduced echelon form shown in **a** looks like:

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

According to Remark 1, this system is inconsistent and has no solution.

iii. $b_1 = 0, b_2 = 1, b_3 = 2$

Answer: $b_3 - b_1 - 2b_2 = 2 - 0 - 2 = 0$. The reduced echelon form shown in **a** looks like:

$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$x_1 = 2x_3 + 3$, $x_2 = x_3 - 1$, and x_3 is the free variable, which can be any number. A particular solution could be: $x_3 = 1, x_1 = 5, x_2 = 0$.