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Math 311

Homework #2

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Section 1.5

Question 32-38: Refer the given matrices and vectors, and find each of the following multiplications.

Answer:

$$DC = \begin{bmatrix} 2 & 1 & 3 & 6 \\ 2 & 0 & 0 & 4 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 8 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 50 & 11 \\ 16 & 10 \\ 3 & -2 \\ 28 & 4 \end{bmatrix}.$$

$$uv = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}. \qquad vu = \begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix} = [14].$$

$$Bu = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \end{bmatrix}.$$

$$CB = \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 8 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & 8 \\ 7 & 12 \\ 5 & 14 \end{bmatrix}.$$

Question: Given the augmented matrix for a system of linear equations. Give the vector form for the general solution.

44. $\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

46. $\begin{bmatrix} 1 & 0 & -1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 3 & 4 & 0 \end{bmatrix}$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Question 59: Let A and B be matrices such that the product AB is defined and is a square matrix. Argue that the product BA is also defined and is a square matrix.

Answer: Let A be a $m_A \times n_A$ matrix, and B be a $m_B \times n_B$ matrix. AB is defined tells us $n_A = m_B$. AB is a square matrix tells us $m_A = n_B$.

Now we look at the product, BA, a $m_B \times n_B$ matrix multiplies with a $m_A \times n_A$ matrix. Since $m_A = n_B$, BA is defined. Since $n_A = m_B$, the product BA is a square matrix.

Question 60: Let A and B be matrices such that the product AB is defined. Use Theorem 6 to prove each of the following.

a. If B has a column of zeros, then so does AB.

Answer: According to Theorem 6, the jth column of AB is AB_j, where B_j is the corresponding column in B. Here, we have B_j is a zero column, so the problem is to prove $AB_{zero} = 0$.

According the Definition 8, the expression of each item in AB_{zero} can be written as $\sum_{k=0}^{n} A_{ik}B_{kj}$. As for here, all $B_{kj} = 0$, therefore $\sum_{k=0}^{n} A_{ik}B_{kj} = 0$ for all entries of AB_{zero}. Hence, we have a zero column AB_j in AB.

b. If B has two identical columns, then so does AB.

Answer: According to Theorem 6, the jth column of AB is AB_j, where B_j is the corresponding column in B. So the problem is to prove $AB_i = AB_{j'}$, and we have $B_i = B_{j'}$. It is obvious that $AB_i = AB_{j'}$ holds.

Question 67: An $(n \times n)$ matrix $T = (t_{ij})$ is called upper triangular if $t_{ij} = 0$ whenever $i > j$. Suppose that A and B are $(m \times n)$ upper-triangular matrices. Use Definition 8 to prove that the product AB is upper-triangular. That is, show that the ijth entry of AB is zero when $i > j$.

Answer: According the Definition 8, $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$. Expending it we obtain:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

$$= A_{i1} \cdot B_{1j} + A_{i2} \cdot B_{2j} + \cdots + A_{ik} \cdot B_{kj} + \cdots + A_{in} \cdot B_{nj}$$

Looking at each entry that has $i > j$, there are two different cases:

1. if we have $i \le k$, since $j < i$, we would have $j < i \le k$, which is $j < k$. So $B_{kj} = 0$, as B is an upper-triangular matrix.

2. if we have $i > k$, then $A_{ik} = 0$, as A is an upper-triangular matrix.

In either case, the multiplication of $A_{ik} \cdot B_{kj} = 0$. Therefore we've shown that in the formula above, every entry would end up equals to zero. The summation, therefore, is also zero. Up to this point, we've proved that $(AB)_{ij} = 0$ if $i > j$. So, the result matrix of two upper-triangular matrices is also an upper-triangular matrix.

Section 1.6

Question 26: Let A and B be (2×2) matrices. Prove or find a counter example for this statement: $(A - B)(A + B) = A^2 - B^2$.

Answer: Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and B be $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$. By expanding the left side of the equation we get:

$$A - B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a - e & b - f \\ c - g & d - h \end{bmatrix}$$

$$A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}$$

$$(A - B)(A + B) = \begin{bmatrix} a - e & b - f \\ c - g & d - h \end{bmatrix} \cdot \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}$$

$$= \begin{bmatrix} (a-e)(a+e) + (b-f)(c+g) & (a-e)(b+f) + (b-f)(d+h) \\ (a+e)(c-g) + (c+g)(d-h) & (b+f)(c-g) + (d-h)(d+h) \end{bmatrix}.$$

Then we expand the right side of the equation. We have:

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e^2 + fg & ef + fh \\ eg + gh & fg + h^2 \end{bmatrix}$$

$$A^2 - B^2 = \begin{bmatrix} a^2 + bc - e^2 - fg & ab + bd - ef - fh \\ ac + cd - eg - gh & bc + d^2 - fg - h^2 \end{bmatrix}.$$

In order to have the left side of the equation equal to the right side of the equation, we have:

$$(a-e)(a+e) + (b-f)(c+g) = a^2 + bc - e^2 - fg$$

$$(a-e)(b+f) + (b-f)(d+h) = ab + bd - ef - fh$$

$$(a+e)(c-g) + (c+g)(d-h) = ac + cd - eg - gh$$

$$(b+f)(c-g) + (d-h)(d+h) = bc + d^2 - fg - h^2$$

After simplifying these equations, we have:

$$bg - cf = 0$$

$$af + bh - df - be = 0$$

$$ce + dg - ag - ch = 0$$

If the input matrices A and B do not satisfy any of these three equations, $(A-B)(A+B) = A^2 - B^2$ would not hold for them. For example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, bg - cf = 2 \times 3 - 3 \times 3 = -3 \neq 0$:

$$(A-B)(A+B) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -6 & -8 \\ 0 & 0 \end{bmatrix}$$

$$A^2 - B^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \begin{bmatrix} 10 & 15 \\ 15 & 25 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ -6 & -8 \end{bmatrix}.$$

Apparently, $\begin{bmatrix} -6 & -8 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} -2 & -6 \\ -6 & -8 \end{bmatrix}$. Therefore we found the counterexample for the statement: $(A-B)(A+B) = A^2 - B^2$.

Question 27: Let A and B be (2×2) matrices. Such that $A^2 = AB$ and $A \neq \mathcal{O}$. Can we assert that, by cancellation, $A = B$? Explain.

Answer: Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and B be $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$. By expanding the left side and the right side of the equation we get:

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$$AB = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}$$

In order to have the left side of the equation equal to the right side of the equation, we have:

$$a(a - a') + b(c - c') = 0$$

$$a(b - b') + b(d - d') = 0$$

$$c(a - a') + d(c - c') = 0$$

$$c(b - b') + d(d - d') = 0$$

To simplify things, let's have $a - a' = a''; b - b' = b''; c - c' = c''$ and $d - d' = d''$. We obtain the following function:

$$\begin{bmatrix} \mathrm{a}'' & \mathrm{c}'' & 0 & 0 \\ \mathrm{b}'' & \mathrm{d}'' & 0 & 0 \\ 0 & 0 & \mathrm{a}'' & \mathrm{c}'' \\ 0 & 0 & \mathrm{b}'' & \mathrm{d}'' \end{bmatrix} \cdot \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{The augmented form:} \begin{bmatrix} \mathrm{a}'' & \mathrm{c}'' & 0 & 0 & 0 \\ \mathrm{b}'' & \mathrm{d}'' & 0 & 0 & 0 \\ 0 & 0 & \mathrm{a}'' & \mathrm{c}'' & 0 \\ 0 & 0 & \mathrm{b}'' & \mathrm{d}'' & 0 \end{bmatrix}$$

To reduce the augmented matrix we obtain:

$$\begin{bmatrix} a'' & c'' & 0 & 0 & 0 \\ b'' & d'' & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix} \to \frac{1}{a''}R_1 \to \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ b'' & d'' & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to R_2 - b''R_1 \to \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ 0 & (a''d'' - b''c'')/a'' & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to \frac{a''}{(a''d'' - b''c'')}R_2 \to \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to R_1 - \frac{c''}{a''}R_2 \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to \frac{1}{a''}R_3 \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & c''/a'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to R_4 - b''R_3 \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & c''/a'' & 0 \\ 0 & 0 & 0 & (a''d'' - b''c'')/a'' & 0 \end{bmatrix}$$

$$\to \frac{a''}{(a''d'' - b''c'')}R_4 \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & c''/a'' & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\to R_3 - \frac{c''}{a''}R_4 \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If the reduction processes successfully, we have one and only one solution for this linear system: $A = O$, which is explicitly excluded. Therefore, the reduction above can't proceed. As we saw, if $a'' = 0$ or $a''d'' - b''c'' = 0$, the reduction procedure above can't proceed that way.

<u>*Case 1.*</u> $a'' = 0$, $\begin{bmatrix} a'' & c'' & 0 & 0 & 0 \\ b'' & d'' & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix} = \begin{bmatrix} 0 & c'' & 0 & 0 & 0 \\ b'' & d'' & 0 & 0 & 0 \\ 0 & 0 & 0 & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix} = \begin{bmatrix} b'' & d'' & 0 & 0 & 0 \\ 0 & c'' & 0 & 0 & 0 \\ 0 & 0 & b'' & d'' & 0 \\ 0 & 0 & 0 & c'' & 0 \end{bmatrix}$,

it would still end up with $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ under additional conditions. So we are not sure.

<u>*Case 2.*</u> $a''d'' - b''c'' = 0$, we have:

$$\begin{bmatrix} a'' & c'' & 0 & 0 & 0 \\ b'' & d'' & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix} \to \frac{1}{a''}R_1 \to \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ b'' & d'' & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to R_2 - b''R_1 \to \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a'' & c'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to \frac{1}{a''}R_3 \to \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c''/a'' & 0 \\ 0 & 0 & b'' & d'' & 0 \end{bmatrix}$$

$$\to R_4 - b''R_3 \to \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c''/a'' & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & c''/a'' & 0 & 0 & 0 \\ 0 & 0 & 1 & c''/a'' & 0 \end{bmatrix}.$$

So we have: $a + \frac{c''}{a''}b = 0$ and $c + \frac{c''}{a''}d = 0$. Together with the prerequisite condition $a''d'' - b''c'' = 0$. We've shown that any two matrix that satisfy these three equations satisfy $A^2 = AB$. The simplified conditions are:

$$a(a - a') + b(c - c') = 0$$

$$c(a - a') + d(c - c') = 0$$

$$(a - a')(d - d') - (b - b')(c - c') = 0\,.$$

For example: if we have $A = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 0 \\ -12 & 1 \end{bmatrix}$. The left side and right side are:

$$A^2 = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} -7 & 0 \\ -12 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}.$$

So $A^2 = AB$ holds and $A \neq B$.

Question 29: Two of the six matrices listed are symmetric. Identify these matrices..

Answer:

$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 6 \end{bmatrix}$ not symmetric.

$B = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix}$ not symmetric.

$C = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 6 & 1 & 3 & 5 \\ 2 & 4 & 2 & 0 \end{bmatrix}$ not symmetric.

$D = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ symmetric.

$E = \begin{bmatrix} 3 & 6 \\ 2 & 3 \end{bmatrix}$ not symmetric.

$F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ symmetric.

$u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ not symmetric.

$v = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ not symmetric.

Question 31: Let A and B be $(n \times n)$ symmetric matrices. Give a necessary and sufficient condition for AB to be symmetric.

Answer: Since A and B are symmetric matrices, we have $A = A^T$ and $B = B^T$. In order to have AB be symmetric, $AB = (AB)^T$. According to Theorem 10, $(AB)^T = B^T A^T$, therefore we have:

$$AB = (AB)^T = B^T A^T = BA$$

So, if and only if $AB = BA$ the multiplication AB of two symmetric matrices A and B can also be a symmetric matrix.

Question 43: Let $A = \begin{bmatrix} 4 & -2 & 2 \\ 2 & 4 & -4 \\ 1 & 1 & 0 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

a. Verify that $Au = 2u$.

Answer: $Au = \begin{bmatrix} 4 & -2 & 2 \\ 2 & 4 & -4 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$ and $2u = 2\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$. So $Au = 2u$.

b. Without forming A^5, calculate the vector A^5u.

Answer: $A^5u = A^4Au = A^4 2u = 2A^3Au = 2A^3 2u = 4A^2Au = 4A^2 2u = 8AAu = 8A2u = 16Au = 32u.$

$$A^5u = 32u = \begin{bmatrix} 32 \\ 96 \\ 64 \end{bmatrix}.$$

c. Give a formula for $A^n u$, where n is a positive integer. What property from Theorem 8 is required to derive the formula?

Answer: As shown in b. $A^n u = 2^0 A^{n-1} Au = 2^1 A^{n-2} Au = 2^2 A^{n-3} Au = \cdots = 2^{n-1} A^0 Au = 2^n u$. So the formula is $A^n u = 2^n u$.

$$A^n u = 2^n u = \begin{bmatrix} 2^n \\ 3 \cdot 2^n \\ 2^{n+1} \end{bmatrix}.$$

The procedure of deriving this formula is based on Theorem 8 #2, which allows us to swap position of a matrix in matrix-scalar multiplications.