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Math 311

Homework #3

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Section 1.7

Question: Use Definition 12 to determine whether the given matrix is singular or nonsingular. If a matrix M is singular, given all solutions of $Mx = \theta$.

16. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Answer: According to Definition 12, we need to examine $Ax = \theta$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. The augmented form of this matrix is $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$. Reduce this matrix to reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \to R_2 - 3R_1 \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\to -\frac{1}{2}R_2 \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\to R_1 - 2R_2 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

As we can see, there aren't any free variable existing, and the only solution is $x_1 = x_2 = 0$. Therefore based on Definition 12, we've shown that matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a nonsingular matrix.

18. $C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Answer: According to Definition 12, we need to examine $Cx = \theta$, $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. The augmented form of this matrix is $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$. Reduce this matrix to reduced echelon form:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} \to R_2 - 2R_1 \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\to -\frac{1}{2}R_2 \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\to R_1 - 3R_2 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

As we can see, there aren't any free variable existing, and the only solution is $x_1 = x_2 = 0$. Therefore based on Definition 12, we've shown that matrix $C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ is a nonsingular matrix.

22. $F = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: According to Definition 12, we need to examine $Fx = \theta$, $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$. The augmented form of this matrix is $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ Reduce this matrix to reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \frac{1}{3}R_2 \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\to R_1 - 2R_2 \to \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\to R_1 + \frac{1}{3}R_3 \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\to R_2 - \frac{2}{3}R_3 \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

As we can see, there aren't any free variable existing, and the only solution is $x_1 = x_2 = x_3 = 0$. Therefore based on Definition 12, we've shown that matrix $F = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is a nonsingular matrix.

24. $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

Answer: According to Definition 12, we need to examine $Ex = \theta$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$. The augmented form of this matrix is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$ Reduce this matrix to reduced echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \rightarrow R_3 - R_1 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\rightarrow \frac{1}{2} R_2 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\rightarrow R_3 - 3R_2 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As we can see, there is one free variable, x_1. The solutions for this system are $x_2 = 0, x_3 = 0$ and x_1 is arbitrary. Therefore based on Definition 12, we've shown that matrix $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ is a singular matrix.

Question 50: If the set $\{v_1, v_2, v_3\}$ of vectors in R^m is linearly dependent, then argue that the set $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent for every choice of v_4 in R^m.

Answer: According the definition of linearly dependent, A linear combination of the vectors in set $\{v_1, v_2, v_3\}$ equals 0 without requiring all coefficients to be 0's. In other words, there exist non-trivial solutions for $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$. Let's assume $\alpha_1 = x, \alpha_2 = y, \alpha_3 = z$ satisfy the equation, and not all $x, y,$ and z are 0's. It means we have $xv_1 + yv_2 + zv_3 = 0$.

Now we are examining whether or not the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. If we can find a non-trivial solution for $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$, then the set is linearly dependent. If the only solution for this linear combination is $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, then the set is linearly independent.

Obviously $xv_1 + yv_2 + zv_3 + 0v_4 = 0$ holds, and not all the coefficients are 0's. Therefore we found a non-trivial solution $\alpha_1 = x, \alpha_2 = y, \alpha_3 = z, \alpha_4 = 0$ for $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$. For this combination of $\alpha_1, \alpha_2, \alpha_3,$ and α_4 the choice of vector v_4 in R^m does not affect. The set $\{v_1, v_2, v_3, v_4\}$ is, therefore, linearly dependent for any choice of v_4 in R^m.

Question 4: Verify that B is the inverse of A by showing that $AB = BA = I$.

Answer: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 2-2+0 & 0+1+0 & 0+0+0 \\ 3-8+5 & 0+4-4 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ -2+2+0 & 0+1+0 & 0+0+0 \\ 5-8+3 & 0-4+4 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Question: Reduce $[A|I]$ to find A^{-1}. Check solutions by multiplying the given matrix by the derived inverse.

14. $\begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$

Answer: $A = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}, [A|I] = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 6 & 7 & 0 & 1 \end{bmatrix}$

$$[A|I] = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 6 & 7 & 0 & 1 \end{bmatrix}$$

$$\to \frac{1}{2}R_1 \to \begin{bmatrix} 1 & 1.5 & 0.5 & 0 \\ 6 & 7 & 0 & 1 \end{bmatrix}$$

$$\to R_2 - 6R_1 \to \begin{bmatrix} 1 & 1.5 & 0.5 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix}$$

$$\to -\frac{1}{2}R_2 \to \begin{bmatrix} 1 & 1.5 & 0.5 & 0 \\ 0 & 1 & 1.5 & -0.5 \end{bmatrix}$$

$$\to -R_1 - \frac{3}{2}R_2 \to \begin{bmatrix} 1 & 0 & -1.75 & 0.75 \\ 0 & 1 & 1.5 & -0.5 \end{bmatrix}.$$

Therefore $A^{-1} = \begin{bmatrix} -1.75 & 0.75 \\ 1.5 & -0.5 \end{bmatrix}$. To verify this solution:

$$A.A^{-1} = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} . \begin{bmatrix} -1.75 & 0.75 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

18. $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 7 \end{bmatrix}$

Answer: $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 7 \end{bmatrix}, [A|I] = \begin{bmatrix} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{bmatrix}$

$$[A|I] = \begin{bmatrix} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow R_1 - 3R_2 \rightarrow \begin{bmatrix} 1 & 0 & -7 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow R_3 - 2R_2 \rightarrow \begin{bmatrix} 1 & 0 & -7 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{bmatrix}$$

$$\rightarrow -R_3 \rightarrow \begin{bmatrix} 1 & 0 & -7 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}$$

$$\rightarrow R_1 + 7R_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 11 & -7 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}$$

$$\rightarrow R_2 - 4R_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 11 & -7 \\ 0 & 1 & 0 & 0 & -7 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}.$$

Therefore $A^{-1} = \begin{bmatrix} 1 & 11 & -7 \\ 0 & -7 & 4 \\ 0 & 2 & -1 \end{bmatrix}$. To verify this solution:

$$A.A^{-1} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 7 \end{bmatrix} . \begin{bmatrix} 1 & 11 & -7 \\ 0 & -7 & 4 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Question 50: Find the (3×3) nonsingular matrix A if $A^2 = AB + 2A$, where

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

Answer: Since matrix A is a nonsingular matrix, there exists the inverse matrix A^{-1}, such that $A.A^{-1} = I$. The identity matrix here is a (3×3) matrix because A is (3×3).

$$A^2 = AB + 2A$$

$$\Longrightarrow A^{-1} \cdot A^2 = A^{-1} \cdot (AB + 2A)$$

$$\Longrightarrow A^{-1} \cdot A \cdot A = A^{-1} \cdot AB + 2A^{-1} \cdot A$$

$$\Longrightarrow I \cdot A = I \cdot B + 2 \cdot I$$

$$\Longrightarrow A = B + 2 \cdot I$$

$$\Longrightarrow A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ -1 & 4 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix}.$$

Finally, we double check the solution by plugging it in the given equation:

$$A^2 = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 5 & -5 \\ -2 & 33 & 16 \\ -7 & 31 & 18 \end{bmatrix}$$

$$AB + 2A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ -1 & 4 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 5 & -5 \\ -2 & 33 & 16 \\ -7 & 31 & 18 \end{bmatrix}.$$

Left side equals right side. The solution of matrix A is $\begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix}$.