Student: Yu Cheng (Jade)

Math 311

Homework #5

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Section 3.3

Question: Refer to the following vectors either show that $Sp(S) = R^3$ or give an algebraic specification for $Sp(S)$. If $Sp(S) \neq R^3$, then give a geometric description of $Sp(S)$.

$$v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$y = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \qquad z = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

12. $S = \{v\}$

Answer: $Sp(S) = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ a \text{ is any real number} \right\}$. This is a line in R^3 through the origin and $(1, 2, 0)$. The equivalence equations for this line are: $x = t, y = 2t, z = 0$. This is derived as below:

$$\begin{bmatrix} 1 & b_1 \\ 2 & b_2 \\ 0 & b_3 \end{bmatrix} \rightarrow R_2 - 2R_1 \rightarrow \begin{bmatrix} 1 & b_1 \\ 0 & b_2 - 2b_1 \\ 0 & b_3 \end{bmatrix}$$

$$\Rightarrow b_1 \text{ is any real number}, b_2 = 2b_1, b_3 = 0 \,.$$

14. $S = \{v, w\}$

Answer: $Sp(S) = \left\{ x \in R^3 : x = a_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},\ a_1 \text{and } a_2 \text{ are any real numbers} \right\}$. This is a plane in R^3 covering the origin, $(1, 2, 0)$, and $(0, -1, 1)$. The equivalence equations for this plan is: $y - 2x + z = 0$. This is derived as below:

$$\begin{bmatrix}1 & 0 & b_1\\ 2 & -1 & b_2\\ 0 & 1 & b_3\end{bmatrix} \to R_2 \leftrightarrow R_3 \to \begin{bmatrix}1 & 0 & b_1\\ 0 & 1 & b_3\\ 2 & -1 & b_2\end{bmatrix}$$

$$\to R_3 - 2R_1 \to \begin{bmatrix}1 & 0 & b_1\\ 0 & 1 & b_3\\ 0 & -1 & b_2 - 2b_1\end{bmatrix}$$

$$\to R_3 + R_2 \to \begin{bmatrix}1 & 0 & b_1\\ 0 & 1 & b_3\\ 0 & 0 & b_2 - 2b_1 + b_3\end{bmatrix}$$

$$\Rightarrow b_2 - 2b_1 + b_3 = 0\,.$$

16. $S = \{v, w, x\}$

Answer: $Sp(S) = \left\{x \in R^3 : x = a_1\begin{bmatrix}1\\2\\0\end{bmatrix} + a_2\begin{bmatrix}0\\-1\\1\end{bmatrix} + a_3\begin{bmatrix}1\\1\\-1\end{bmatrix},\ a_1, a_2, \text{and } a_3 \text{ are any real numbers}\right\}$. In other words, $Sp(S) = R^3$. This is derived as below:

$$\begin{bmatrix}1 & 0 & 1\\ 2 & -1 & 1\\ 0 & 1 & -1\end{bmatrix} \to R_2 \leftrightarrow R_3 \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}$$

$$\to R_3 - 2R_1 \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & -1 & -1\end{bmatrix}$$

$$\to R_3 + R_2 \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & -2\end{bmatrix}$$

$$\to -\frac{1}{2}R_3 \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

$$\to R_2 + R_3 \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\to R_1 - R_3 \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow v, x,$ and y are linearly indipendent vectors

$\Rightarrow Sp(S) = R^3$.

Question 20: Let S be the set given as $S = \{v, w\} = \left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$. For each vector given below, determine whether the vector is in $Sp(S)$. Express those vectors that are in $Sp(S)$ as a linear combination of v and w.

a. $\begin{bmatrix}1\\1\\1\end{bmatrix}$ **b.** $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ **c.** $\begin{bmatrix}1\\2\\0\end{bmatrix}$

d. $\begin{bmatrix}2\\3\\1\end{bmatrix}$ **e.** $\begin{bmatrix}-1\\2\\4\end{bmatrix}$ **f.** $\begin{bmatrix}1\\1\\3\end{bmatrix}$

Answer: $Sp(S) = \left\{x \in R^3 : x = a_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + a_2 \begin{bmatrix}0\\-1\\1\end{bmatrix},\ a_1 \text{and } a_2 \text{ are any real numbers}\right\}$. This is a plane in R^3 covering the origin, $(1, 2, 0)$, and $(0, -1, 1)$. The equivalence equations for this plan is: $y - 2x + z = 0$.

a. $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is on this plane because $1 - 2 \times 1 + 1 = 0$. $\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix}$, so $a_1 = a_2 = 1$.

b. $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ is not on this plane because $1 - 2 \times 1 - 1 = -2 \neq 0$.

c. $\begin{bmatrix}1\\2\\0\end{bmatrix}$ is on this plane because $2 - 2 \times 1 + 0 = 0$. $\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}0\\-1\\1\end{bmatrix}$, so $a_1 = 1, a_2 = 0$.

d. $\begin{bmatrix}2\\3\\1\end{bmatrix}$ is on this plane because $3 - 2 \times 2 + 1 = 0$. $\begin{bmatrix}2\\3\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}1\\2\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix}$, so $a_1 = 2, a_2 = 1$.

e. $\begin{bmatrix}-1\\2\\4\end{bmatrix}$ is not on this plane because $2 - 2 \times (-1) + 4 = 8 \neq 0$.

f. $\begin{bmatrix}1\\1\\3\end{bmatrix}$ is not on this plane because $1 - 2 \times 1 + 3 = 2 \neq 0$.

Question: Give an algebraic specification for the null space and the range of the given matrix A.

30. $A = \begin{bmatrix}1 & -1 & 2\\2 & -1 & 5\end{bmatrix}$

Answer: Reduce the matrix we have:

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 5 \end{bmatrix} \to R_2 - 2R_1 \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\to R_1 + R_2 \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Null space: The kernel is $\mathcal{N}(A) = \left\{ x \in R^3 \colon x = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix},\ x_3 \text{ is any real number} \right\}$. This is a line in R^3 through the origin and $(-3, -1, 1)$.

Range: The column space can be expressed as a linear combination of the independent column vectors: $Sp(A) = \left\{ x \in R^2 \colon x = a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ a_1 \text{and } a_2 \text{ are any real numbers} \right\}$ which is the same as $Sp(A) = R^2$.

32. $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 1 & 5 \end{bmatrix}$

Answer: Reduce the augmented matrix we have:

$$\begin{bmatrix} 1 & 3 & b_1 \\ 2 & 7 & b_2 \\ 1 & 5 & b_3 \end{bmatrix} \to R_2 - 2R_1 \to \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 1 & 5 & b_3 \end{bmatrix}$$

$$\to R_3 - R_1 \to \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 2 & b_3 - b_1 \end{bmatrix}$$

$$\to R_1 - 3R_2 \to \begin{bmatrix} 1 & 0 & 7b_1 - 3b_2 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 2 & b_3 - b_1 \end{bmatrix}$$

$$\to R_3 - 2R_2 \to \begin{bmatrix} 1 & 0 & 7b_1 - 3b_2 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + 3b_1 - 2b_2 \end{bmatrix}.$$

Null space: The kernel is $\mathcal{N}(A) = \left\{ x \in R^2 \colon x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. In other words, $\mathcal{N}(A) = \{\theta\}$.

Range: The column space can be expressed as a linear combination of the independent column vectors: $Sp(A) = \left\{ x \in R^3 \colon x = a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix},\ a_1 \text{and } a_2 \text{ are any real numbers} \right\}$ which can also be expressed as $z + 3x - 2y = 0$. It is a plane in R^3 covering the origin, $(1, 2, 1)$, and $(3, 7, 5)$.

Question 50: Identify the range and the null space for each of the following.

a. The $(n \times n)$ identity matrix

Answer: *Null space:* There's no free variable in an identity matrix. We can get the zero matrix only by multiplying I with the zero vector. Therefore the null space is

$$\mathcal{N}(A) = \left\{ x \in R^{\mathrm{n}} : x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow \mathcal{N}(A) = \{\theta\}\,.$$

Range: The column space can be expressed as a linear combination of the independent column vectors:

$$Sp(A) = \left\{ x \in R^n : x = a_1 \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, a_1, a_2, \cdots, a_n, \text{are any real numbers} \right\}$$

$$\Rightarrow Sp(A) = R^n\,.$$

b. The $(n \times n)$ zero matrix

Answer: *Null space:* They are all free variables in the zero matrix. We can get the zero matrix by multiplying θ with the any vector. Therefore the null space is:

$$\mathcal{N}(A) = R^n\,.$$

Range: The column space can be expressed as a linear combination of the independent column vectors, since they are all zero vectors, the column vectors span only the origin.

$$Sp(A) = \left\{ x \in R^n : x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow Sp(A) = \{\theta\}\,.$$

c. The $(n \times n)$ nonsingular matrix

Answer: *Null space:* By definition $Sx = \theta$ has only one solution for nonsingular matrix S. The solution is $x = \theta$. Therefore the null space is:

$$\mathcal{N}(A) = \left\{ x \in R^{\mathrm{n}} : x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow \mathcal{N}(A) = \{\theta\}\,.$$

Range: The nonsingular matrix can be reduced to an identical matrix. The column space is therefore:

$$Sp(A) = R^n .$$

Question 52: Let A be an $(m \times r)$ matrix and B an $(r \times n)$ matrix.

a. Show that $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$.

Answer: Assuming vector $x \in \mathcal{N}(B)$, which means $Bx = \theta$, we would have $ABx = A(Bx) = A\theta = \theta$. So vector $x \in \mathcal{N}(AB)$ as well. Therefore, we've shown that any vector belongs to $\mathcal{N}(B)$ also belongs to $\mathcal{N}(AB)$.Hence, we have relation, $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$

b. Show that $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$.

Answer: Assuming vector $y \in \mathcal{R}(AB)$ which means $ABx = y$, we would have $ABx = A(Bx) = Ax' = y$. So vector $y \in \mathcal{R}(A)$ as well. Therefore, we've shown that any vector belongs to $\mathcal{R}(AB)$ also belongs to $\mathcal{R}(A)$. Hence, we have relation, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$.