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**Math 311**

**Homework #6**

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## Section 3.4

**Question:** Let $W$ be the subspace of $R^4$ consisting of vectors of the form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Find a basis for $W$ when the components of $x$ satisfy the given conditions.

**2.** $x_1 + x_2 - x_3 + x_4 = 0$

$x_2 - 2x_3 - \mathrm{x}_4 = 0$

**Answer:** If we consider $x_3$ and $x_4$ as the free variables, then we can express the other two variables as, $x_2 = 2x_3 + x_4$, $x_1 = x_3 - x_4 - 2x_3 - x_4 = -x_3 - 2x_4$. Therefore we have the following expression for the solution of this system:

$$x = \begin{bmatrix} -x_3 - 2x_4 \\ 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence we've shown $W$ can be expressed as a linear combination of the two vectors listed above.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis of the system.}$$

**4.** $x_1 - x_2 + x_3 = 0$

**Answer:** If we consider $x_1$ and $x_2$ as the free variables, we can express $x_3$ as $-x_1 + x_2$. $x_4$ is obviously a free variable, since there's no restriction on the value of $x_4$. Therefore we have the following expression for the solution of this system:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ -x_1 + x_2 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence we've shown $W$ can be expressed as a linear combination of the three vectors listed above

$$\left\{\begin{bmatrix}1\\0\\-1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \text{ is a basis of the system.}$$

**Question 10:** Let $W$ be the subspace described in Exercise 2. For each vector $x$ that follows, determine whether $x$ is in $W$. If $x$ is in $W$, then express $x$ as a linear combination of the basis vectors found in Exercise 2.

**a.** $x = \begin{bmatrix}-3\\3\\1\\1\end{bmatrix}$

**Answer:** Plugging in $x_1 + x_2 - x_3 + x_4 = 0$, we have $-3 + 3 - 1 + 1 = 0$. Plugging in $x_2 - 2x_3 - x_4 = 0$, we have $3 - 2 \times 1 - 1 = 0$. So, vector $x$ is in $W$.

$$x = \begin{bmatrix}-3\\3\\1\\1\end{bmatrix} = 1 \times \begin{bmatrix}-1\\2\\1\\0\end{bmatrix} + 1 \times \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}.$$

**b.** $x = \begin{bmatrix}0\\3\\2\\-1\end{bmatrix}$

**Answer:** Plugging in $x_1 + x_2 - x_3 + x_4 = 0$, we have $0 + 3 - 2 - 1 = 0$. Plugging in $x_2 - 2x_3 - x_4 = 0$, we have $3 - 2 \times 2 + 1 = 0$. So, vector $x$ is in $W$.

$$x = \begin{bmatrix}0\\3\\2\\-1\end{bmatrix} = 2 \times \begin{bmatrix}-1\\2\\1\\0\end{bmatrix} + (-1) \times \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}.$$

**c.** $x = \begin{bmatrix}7\\8\\3\\2\end{bmatrix}$

**Answer:** Plugging in $x_1 + x_2 - x_3 + x_4 = 0$, we have $7 + 8 - 3 + 2 = 14 \neq 0$. So, vector $x$ is not in $W$ and can't be expressed as a linear combination of the basis vectors of $W$.

**d.** $x = \begin{bmatrix} 4 \\ -2 \\ 0 \\ -2 \end{bmatrix}$

**Answer:** Plugging in $x_1 + x_2 - x_3 + x_4 = 0$, we have $4 - 2 - 0 - 2 = 0$. Plugging in $x_2 - 2x_3 - x_4 = 0$, we have $-2 - 2 \times 0 + 2 = 0$. So, vector $x$ is in $W$.

$$x = \begin{bmatrix} 4 \\ -2 \\ 0 \\ -2 \end{bmatrix} = 0 \times \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + (-2) \times \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

**Question 12:** Matrix $A$ is given as $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$.

**a.** Find a matrix $B$ in reduced echelon form such that $B$ is a raw equivalency of $A$.

**Answer:** $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \rightarrow R_2 - R_1 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & 3 & 5 \end{bmatrix}$

$$\rightarrow R_3 - 2R_1 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\rightarrow R_3 \leftrightarrow R_2 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow R_1 - R_2 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We obtain matrix above from $A$ by taking row operations. It is, therefore, a row equivalency of $A$.

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

**b.** Find a basis for the null space of $A$.

**Answer:** From the solution for the previous question, we know that the reduced augmented matrix of the equation $Ax = \theta$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore we have the null space of $A$:

$$\mathcal{N}(A) = \left\{ x \in R^3 \colon x = a \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, a \text{ is any real number} \right\}.$$

From the expression above, it's clear that a basis for the null space of $A$ is $\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$.

**c.** Find a basis for the range of $A$ that consists the columns of $A$. For each column of $A$, $A_j$ that does not appear in the basis, express $A_j$ as a linear combination of the basis vectors.

**Answer:** From the solution for the sub question **a**, we've learned that there are two linearly independent column vectors in $A$. If we adapt the form of matrix $B$, then the first two column vectors are linearly independent and can be used to form a basis for the range of $A$.

$$Sp(A) = \left\{x \in R^3 : x = a_1\begin{bmatrix}1\\1\\2\end{bmatrix} + a_2\begin{bmatrix}1\\1\\3\end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers}\right\}.$$

$$\Rightarrow \left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}\right\} \text{ is a basis for the range of } A\,.$$

The third column vector is, therefore, dependent and can be expressed as a linear combination of the first two vectors:

$$\begin{bmatrix}2\\2\\5\end{bmatrix} = 1 \times \begin{bmatrix}1\\1\\2\end{bmatrix} + 1 \times \begin{bmatrix}1\\1\\3\end{bmatrix}.$$

**c.** Exhibit a basis for the row space of $A$.

**Answer:** From the solution for the sub question **a**, we've learned that there are two linearly independent row vectors in $A$. If we adapted the form of matrix $B$, then the first two row vectors are linearly independent and can be used to form a basis for the row space of $A$.

$$RowSpace(A) = \left\{x \in R^3 : x = a_1\begin{bmatrix}1\\0\\1\end{bmatrix} + a_2\begin{bmatrix}0\\1\\1\end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers}\right\}.$$

$$\Rightarrow \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} \text{ is a basis for the row space of } A\,.$$

**Question 18:** Obtain a basis for the range of $A$ using the technique of Example 7 in the textbook. Matrix $A$ is given as: $A = \begin{bmatrix}1 & 1 & 2\\1 & 1 & 2\\2 & 3 & 5\end{bmatrix}$.

**Answer:** the range of $A$ can be viewed as the row space of the matrix $A^T$, where

$$A^T = \begin{bmatrix}1 & 1 & 2\\1 & 1 & 3\\2 & 2 & 5\end{bmatrix} \to R_2 - R_1 \to \begin{bmatrix}1 & 1 & 2\\0 & 0 & 1\\2 & 2 & 5\end{bmatrix}$$

$$\to R_3 - 2R_1 \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to R_1 - 2R_2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to R_3 - R_2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B^T \,.$$

The nonzero rows of the reduced matrix above form a basis of the row space of $A^T$. Consequently the nonzero columns of

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Are a basis for the span of $A$. Specifically, the solution is as below:

$$Sp(A) = \left\{ x \in R^3 : x = a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers} \right\}.$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the range of } A\,.$$

**Question 22:** Given the set $S$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$$

**a.** Find a subset of $S$ that is a basis for $Sp(S)$ using the technique illustrated in Example 6.

**Answer:** We need to solve the dependence relation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = \theta$$

And then determine which if the $v_j$'s can be eliminated. If $V$ is the $(2 \times 3)$ matrix

$$V = [v_1, v_2, v_3]$$

Then the augmented matrix $[V|\theta]$ reduces to

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix} \to R_2 - 2R_1 \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -4 & 0 \end{bmatrix}$$

$$\to -\frac{1}{3} R_2 \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 4/3 & 0 \end{bmatrix}$$

$$\to R_1 - 2R_2 \to \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 4/3 & 0 \end{bmatrix}.$$

The system of equations with augmented matrix above has solution

$$x_1 = \frac{1}{3}x_3$$

$$x_2 = \frac{4}{3}x_3$$

Where $x_3$ is unconstrained variable. In particular, the set $S$ is linearly dependent. Moreover, taking $x_3 = 1$ yields $x_1 = 1/3,\ x_2 = 4/3$. Thus the linear combination $x_1v_1 + x_2v_2 + x_3v_3 = \theta$ becomes

$$\frac{1}{3}v_1 + \frac{4}{3}v_2 + v_3 = \theta$$

$$\Rightarrow v_3 = -\frac{1}{3}v_1 - \frac{4}{3}v_2 .$$

It follows that $v_3$ is redundant and can be removed from the spanning set. The set $\{v_1, v_2\}$ is a linearly independent set and therefore a basis of the span of $S$.

$$Sp(S) = \left\{x \in R^2 : x = a_1\begin{bmatrix}1\\2\end{bmatrix} + a_2\begin{bmatrix}2\\1\end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers}\right\}$$

$$= R^2$$

$$\Rightarrow \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\} \text{ is a basis for the range of } S .$$

**b.** Find a basis for $SP(S)$ using the technique illustrated in Example 7

**Answer:** Let $A = SP(S)$. The range of $A$ can be viewed as the row space of the matrix $A^T$, where

$$A^T = \begin{bmatrix}1 & 2\\2 & 1\\3 & 2\end{bmatrix} \to R_2 - 2R_1 \to \begin{bmatrix}1 & 2\\0 & -3\\3 & 2\end{bmatrix}$$

$$\to R_3 - 3R_1 \to \begin{bmatrix}1 & 2\\0 & -3\\0 & -4\end{bmatrix}$$

$$\to -\frac{1}{3}R_2 \to \begin{bmatrix}1 & 2\\0 & 1\\0 & -4\end{bmatrix}$$

$$\to R_1 - 2R_2 \to \begin{bmatrix}1 & 0\\0 & 1\\0 & -4\end{bmatrix}$$

$$\to R_3 + 4R_2 \to \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} = B^T .$$

The nonzero rows of the reduced matrix above form a basis of the row space of $A^T$. Consequently the nonzero columns of

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Are a basis for the span of $S$. Specifically, the solution is as below:

$$Sp(S) = \left\{x \in R^2 : x = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers}\right\}$$

$$= R^2$$

$$\Rightarrow \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \text{ is a basis for the range of } S.$$

Since the span of the given vectors is the entire $R^2$, any two independent vectors in $R^2$ form a basis of $SP(S)$.

## Section 3.5

**Question:** Determine by inspection why the given set $S$ is not a basis for $R^2$. (That is, either $S$ is linearly dependent or $S$ does not span $R^2$.)

$u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad$ $u_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad$ $u_5 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

**2.** $S = \{u_2\}$

**Answer:** The given set $S$ is not a basis for $R^2$ because the vector given in the set $S$ does not span $R^2$. This conclusion is based on Theorem 9-2.

Theorem 9-2 tells us any set of fewer than $p$ vectors in $W$ does not span $W$, where $W$ is a subspace of $R^n$ with $\dim(W) = p$. In this case, $R^2$ is $W$, and the dimension of $R^2$ is of course $\dim(R^2) = 2 = p$. While we have only one vector in set $S$, because $1 < 2$, set $S$ does not span $R^2$ and, therefore, can't be a basis of $R^2$.

**4.** $S = \{u_2,\ u_3,\ u_5\}$

**Answer:** The given set $S$ is not a basis for $R^2$ because $S$ is not linearly independent. This conclusion is based on Theorem 9-1.

Theorem 9-1 tells us any set of $p+1$ or more vectors in $W$ is linearly dependent, where $W$ is a subspace of $R^n$ with $\dim(W)=p$. In this case, $R^2$ is $W$, and the dimension of $R^2$ is of course $\dim(R^2)=2=p$. While we have only three vectors in set $S$, because $3 \geq 2+1$ holds, set $S$ is linearly dependent and, therefore, can't be a basis of $R^2$.

**Question:** Use Theorem 9, property 3, to determine whether the given set is a basis for the indicated vector space.

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

**12.** $S = \{v_1, v_2, v_3\}$ for $R^3$

**Answer:** Theorem 9-3 tells us that any set of $p$ vectors that spans $W$ is a basis for $W$. In this case, $p=3$, and we have three vectors in $S$. So we just need to determine if the three vectors in $S$ are linearly independent.

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \to R_2 + R_1 \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\to R_3 - R_1 \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\to R_3 - 2R_2 \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\to -R_3 \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to R_1 - R_3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The column vectors of the matrix above are the vectors in set $S$, and the column vectors are shown to be linearly independent. Therefore $S$ satisfies the conditions for Theorem 9-3, $S = \{v_1, v_2, v_3\}$ is a basis $R^3$.

**14.** $S = \{ v_2, v_3, v_4\}$ for $R^3$

**Answer:** Theorem 9-3 tells us that any set of $p$ vectors that spans $W$ is a basis for $W$. In this case, $p=3$, and we have three vectors in $S$. So we just need to determine if the three vectors in $S$ are linearly independent.

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 3 \end{bmatrix} \to R_2 \leftrightarrow R_1 \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$\to R_3 - 2R_1 \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix}$$

$$\to R_1 + R_2 \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix}$$

$$\to R_3 - 2R_2 \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\to -R_3 \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to R_1 - 2R_3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to R_2 + R_3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The column vectors of the matrix above are the vectors in set $S$, and the column vectors are shown to be linearly independent. Therefore $S$ satisfies the conditions for Theorem 9-3, $S = \{v_2, v_3, v_4\}$ is a basis $R^3$.

**Question 22:** Find a basis for $\mathcal{N}(A)$ and give the nullity and the rank of $A$

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

**Answer:** Reduce the augmented matrix of $A$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -5 & 1 \end{bmatrix} \to -R_1 \to \begin{bmatrix} 1 & -2 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

$$\to R_2 - 2R_1 \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\to -R_2 \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\to R_1 + 2R_2 \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}.$$

There is one free variable $x_3$. Since the kernel space can be expresses as a linearly combination of one vector, this vector itself forms a basis of $\mathcal{N}(A)$.

$$\mathcal{N}(A) = \left\{ x \in R^3 : x = a \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, a \text{ is any real number} \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the null space of } A, \text{and } nullity(A) = 1 \,.$$

From the reduced form of the original matrix, we learned that there are two independent column vectors , and they span the entire $R^2$.

$$Sp(A) = \left\{ x \in R^2 : x = a_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers} \right\}$$

$$= R^2$$

$$\Rightarrow \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\} \text{ is a basis for the range of } A, \text{and } rank(A) = 2 \,.$$

Of course, any two independent vectors in $R^2$ form a basis of $Sp(A)$ in this case. Also notice $rank(A) + nullity(A) = n$, that we can derive $rank(A)$ directly. $rank(A) = n - nullity(A) = 3 - 1 = 2$.

**Question 26:** Find a basis for $\mathcal{R}(A)$ and give the nullity and the rank of $A$

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

**Answer:** Reduce the augmented matrix of $A$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix} \to R_2 - 2R_1 \to \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

$$\to R_3 - 2R_1 \to \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\to \frac{1}{2}R_2 \to \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\to R_1 - R_2 \to \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\to R_3 + R_2 \to \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the reduced, we learned that there are two independent column vectors, the first two, and they would form a basis for the range of $A$.

$$Sp(A) = \left\{ x \in R^3 : x = a_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers} \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the range of } A, \text{ and } rank(A) = 2 \ .$$

Since $rank(A) + nullity(A) = n$, we would know directly that $nullity(A) = n - rank(A) = 4 - 2 = 2$. This can also be derived by write out the kernel space of $A$.

$$\mathcal{N}(A) = \left\{ x \in R^4 : x = a_1 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}, a_1 \text{ and } a_2 \text{ are any real numbers} \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the null space of } A, \text{ and } nullity(A) = 2 \ .$$

**Question 32:** Let $U$ and $V$ be subspaces of $R^n$, and suppose that $U$ is a subset of $V$. Prove that $\dim(U) \le \dim(V)$. If $\dim(U) = \dim(V)$, prove that $V$ is contained in $U$, and thus conclude that $U = V$.

**Answer:** Let's say $\dim(V) = q$. Let's assume $\dim(U) > \dim(V)$, we would have $\dim(U) \ge q + 1$, since $\dim(V) = q$. It means the basis of $U$ consist of a minimal of $q + 1$ vectors. We also know that $U$ is a subset of $V$, so all vectors in $U$ are also in $V$. According to Theorem 9-1, any set of $p + 1$ or more vectors in $W$ is linearly dependent, where $\dim(W) = p$, the $q + 1$ vectors in $V$ are, therefore, linearly depended since $\dim(V) = q$. This is a conflict because vectors in a basis are linearly independent by definition. Therefore our assumption was wrong. $\dim(U) \le \dim(V)$.

If $\dim(U) = \dim(V) = q$, then $U, V$ have basis consisting $q$ vectors. Considering $q$ vectors that make up basis for $U$. According to Theorem 9-3, any set of $p$ linearly independent vectors in $W$ is a basis for $W$, these $q$ vectors is also a basis for $V$. Therefore any vector in $V$ can be expressed as a linearly combination of the basis vectors for $U$. This is saying that $V \subseteq U$. Since $U$ is known to be a subset of $V$, we have $U = V$.

**Question 40:** Let $A$ be an $(m \times m)$ nonsingular matrix, and let $B$ be an $(m \times n)$ matrix. Prove that $\mathcal{N}(AB) = \mathcal{N}(B)$ and conclude that $rank(AB) = rank(B)$.

**Answer:** Assume vector $v \in \mathcal{N}(AB)$, it means $AB \cdot v = \theta$. We can multiply $A^{-1}$ on both sides of this equation.

$$AB \cdot v = \theta$$

$$\Rightarrow A^{-1}AB \cdot v = A^{-1}\theta$$

$$\Rightarrow IB \cdot v = \theta$$

$$\Rightarrow B \cdot v = \theta$$

$$\Rightarrow v \in \mathcal{N}(B)\,.$$

So we've shown that any vector in the null space of $AB$, $\mathcal{N}(AB)$, is also in the null space of $B$, $\mathcal{N}(B)$. We can prove the other way around using the same approach. Assume vector $u \in \mathcal{N}(B)$, it means $B \cdot u = \theta$. We can multiply $A$ on both sides of this equation.

$$B \cdot u = \theta$$

$$\Rightarrow AB \cdot u = A\theta$$

$$\Rightarrow AB \cdot u = \theta$$

$$\Rightarrow u \in \mathcal{N}(AB)\,.$$

Therefore we've proven any vector in $\mathcal{N}(AB)$ is also in $\mathcal{N}(B)$, and any vector in $\mathcal{N}(B)$ is also in $\mathcal{N}(AB)$. This is the same as saying $\mathcal{N}(AB) = \mathcal{N}(B)$. According to the Remark that if $A$ is an $(m \times n)$ matrix, then $n = rank(A) + nullity(A)$, we have the following. Note that both $AB$ and $B$ are $(m \times n)$ matrices.

$$\mathcal{N}(AB) = \mathcal{N}(B)$$

$$\Rightarrow nullity(AB) = nullity(B)$$

$$\Rightarrow n - rank(AB) = n - rank(B)$$

$$\Rightarrow rank(AB) = rank(B)\,.$$