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Math 311

Homework #7

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Section 3.6

Ex 4: Verify that $\{u_1, u_2, u_3\}$ is an orthogonal set for the given vectors.

$u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ $\quad u_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

Answer: To verify that $\{u_1, u_2, u_3\}$ is an orthogonal set, we would need to verify every pair-wise dot product between the vectors is zero.

$$u_1^T \cdot u_2 = u_2^T \cdot u_1 = 2 + 2 - 4 = 0$$

$$u_1^T \cdot u_3 = u_3^T \cdot u_1 = -4 + 2 + 2 = 0$$

$$u_2^T \cdot u_3 = u_3^T \cdot u_2 = -2 + 4 - 2 = 0$$

According to the definition of orthogonal, set $\{u_1, u_2, u_3\}$ is an orthogonal set.

Ex 10, 12: Express the given vector v in terms of the orthogonal basis $B = \{u_1, u_2, u_3\}$, where u_1, u_2, u_3 are as below:

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad u_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

10: $v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Answer: We have the following equations:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow a - b - c = 0$$

$$a + 2c = 1$$

$$a + b - c = 2$$

Solving this system with three equations and three unknowns a, b, c. We have:

$$a = 1, b = 1, c = 0$$

$$\Rightarrow \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 1 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 0 \times \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

12: $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Answer: We have the following equations:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow a - b - c = 1$$

$$a + 2c = 2$$

$$a + b - c = 1$$

Solving this system with three equations and three unknowns a, b, c. We have:

$$a = \frac{4}{3}, b = 0, c = \frac{1}{3}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \frac{4}{3} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \times \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3} \times \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

Ex 14: Use the *Gram-Schmidt* process to generate an orthogonal set from the given linearly independent vectors given below.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Answer: Follow the steps of *Gram-Schmidt* Algorithm:

$$u_1 = w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$u_2 = w_2 - \frac{u_1^T w_2}{u_1^T u_1} u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \frac{2+0+0+4}{1+0+1+4} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$u_3 = w_3 - \frac{u_1^T w_3}{u_1^T u_1} u_1 - \frac{u_2^T w_3}{u_2^T u_2} u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1+0+0+2}{1+0+1+4} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \frac{1-1+0+0}{1+1+1+0} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \\ 0 \end{bmatrix}$$

We can verify the solution by checking every pair-wise dot product of the derived vectors.

$$u_1^T \cdot u_2 = u_2^T \cdot u_1 = 1 + 0 - 1 + 0 = 0$$

$$u_1^T \cdot u_3 = u_3^T \cdot u_1 = \frac{1}{2} + 0 - \frac{1}{2} + 0 = 0$$

$$u_2^T \cdot u_3 = u_3^T \cdot u_2 = \frac{1}{2} - 1 + \frac{1}{2} + 0 = 0$$

Every pair-wise dot product is zero. The solution is therefore correct $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \\ 0 \end{bmatrix} \right\}$.

Ex 24: The *Cauchy-Schwarz inequality*. Let x and y be vectors in R^n. Prove that $|x^T y| \le \|x\| \|y\|$.

Answer: We start from know inequality, $\|x - cy\|^2 \ge 0$ where $c = x^T y / y^T y$, and $y \ne \theta$.

$$\|x - cy\|^2 = (x - cy)^T (x - cy)$$

$$= \left(x^T - \frac{1}{c} y^T \right) (x - cy)$$

$$= x^T x + y^T y - \frac{1}{c} y^T x - c x^T y$$

$$= \|x\|^2 + \|y\|^2 - \frac{y^T y \cdot y^T x}{x^T y} - \frac{x^T y \cdot x^T y}{y^T y}$$

$y^T x = x^T y$, since they both represent $x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$. Continuing the simplification above we have:

$$\|x - cy\|^2 = \|x\|^2 + \|y\|^2 - \frac{y^T y \cdot y^T x}{x^T y} - \frac{x^T y \cdot x^T y}{y^T y}$$

$$= \|x\|^2 + \|y\|^2 - y^T y - \frac{x^T y \cdot x^T y}{y^T y}$$

$$= \frac{1}{y^T y} \cdot [\|x\|^2 \|y\|^2 + \|y\|^4 - \|y\|^4 - (x^T y)^2]$$

$$= \frac{1}{y^T y}[\|x\|^2\|y\|^2 - (x^T y)^2]$$

$$\geq 0$$

Further simplify the inequality above, we obtain the to-be-proven inequality:

$$\frac{1}{y^T y}[\|x\|^2\|y\|^2 - (x^T y)^2] \geq 0$$

$$\Rightarrow \|x\|^2\|y\|^2 - (x^T y)^2 \geq 0$$

$$\Rightarrow \|x\|^2\|y\|^2 \geq (x^T y)^2$$

$$\Rightarrow \|x\|\|y\| \geq |x^T y\,|.$$

If $y = \theta$, the to-be-proven inequality is simply $\theta = \theta$, $\|x\|\|y\| \geq |x^T y|$ still holds. Therefore we've proven the inequality $\|x\|\|y\| \geq |x^T y\,|$ holds in all cases.

Ex 24: Let $B = \{u_1, u_2, \cdots u_p\}$ be an orthonormal basis for a subspace W. Let v be any vector in W, where $v = a_1u_1 + a_2u_2 + \cdots + a_pu_p$. Show that

$$\|v\|^2 = a_1^2 + a_2^2 + \cdots + a_p^2$$

Answer: By definition $\|v\|^2 = v^T v$. $v = a_1u_1 + a_2u_2 + \cdots + a_pu_p$, and we also have:

$$v^T = \left(a_1u_1 + a_2u_2 + \cdots + a_pu_p\right)^T$$

$$= (a_1u_1)^T + (a_2u_2)^T + \cdots + \left(a_pu_p\right)^T$$

$$= a_1u_1^T + a_2u_2^T + \cdots + a_pu_p^T$$

Therefore $\|v\|^2 = v^T v$ can be expressed as:

$$\|v\|^2 = v^T v$$

$$= \left(a_1u_1 + a_2u_2 + \cdots + a_pu_p\right) \cdot \left(a_1u_1^T + a_2u_2^T + \cdots + a_pu_p^T\right)$$

$$= \left(a_1^2u_1u_1^T + a_1a_2u_1u_2^T + \cdots + a_1a_pu_1u_p^T\right) + \left(a_2^2u_2u_2^T + a_2a_1u_2u_1^T + \cdots + a_2a_pu_pu_p^T\right)$$

$$+ \cdots + \left(a_p^2u_pu_p^T + a_pa_1u_pu_1^T + \cdots + a_pa_{p-1}u_pu_{p-1}^T\right)$$

Since set $B = \{u_1, u_2, \cdots u_p\}$ is be an orthonormal basis for W, every doc product in the form of $u_iu_j^T$, where $i \neq j$, is zero. The equation above is, therefore, simplified to be:

$$\|v\|^2 = (a_1^2u_1u_1^T + 0 + \cdots + 0) + (a_2^2u_2u_2^T + 0 + \cdots + 0) + \cdots + \left(a_p^2u_pu_p^T + 0 + \cdots + 0\right)$$

$$= a_1^2u_1u_1^T + a_2^2u_2u_2^T + \cdots + a_p^2u_pu_p^T$$

$$= a_1^2\|u_1\|^2 + a_2^2\|u_2\|^2 + \cdots + a_p^2\left\|u_p\right\|^2$$

$$= a_1^2 \times 1 + a_2^2 \times 1 + \cdots + a_p^2 \times 1$$

$$= a_1^2 + a_2^2 + \cdots + a_p^2 .$$

Section 3.7

Ex 8-11: Determine whether the function F is a linear transformation

8. $F: R^2 \to R^2$ defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Answer: We need to determine whether the two linearity properties are satisfied by F. Thus let u and v be in R^2,

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Let's verify $F(u + v) = F(u) + F(v)$ first. The left side can be expanded into,

$$F(u + v) = F\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

$$= F\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(u_1 + v_1) - (u_2 + v_2) \\ (u_1 + v_1) + 3(u_2 + v_2) \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1 + 2v_1 - u_2 - v_2 \\ u_1 + v_1 + 3u_1 + 3v_2 \end{bmatrix}.$$

The right side can be expanded into,

$$F(u) + F(v) = \begin{bmatrix} 2u_1 - u_2 \\ u_1 + 3u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 - v_2 \\ v_1 + 3v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1 + 2v_1 - u_2 - v_2 \\ u_1 + v_1 + 3u_1 + 3v_2 \end{bmatrix}.$$

So we've shown that $F(u+v) = F(u) + F(v)$ holds, since the left side equals to the right side. Similarly, we can verify the $F(cu) = cF(u)$, The left hand side can be expanded into,

$$F(cu) = F\left(c\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) = F\left(\begin{bmatrix}cu_1\\cu_2\end{bmatrix}\right) = \begin{bmatrix}2cu_1 - cu_2\\cu_1 + 3cu_2\end{bmatrix}.$$

The right hand side can be expanded into,

$$cF(u) = cF\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) = c\begin{bmatrix}2u_1 - u_2\\u_1 + 3u_2\end{bmatrix} = \begin{bmatrix}2cu_1 - cu_2\\cu_1 + 3cu_2\end{bmatrix}.$$

So we've shown that $F(cu) = cF(u)$ holds, since the left side equals to the right side. Therefore function F is a linear transformation.

9. $F: R^2 \to R^2$ defined by

$$F\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_2\\x_1\end{bmatrix}$$

Answer: We need to determine whether the two linearity properties are satisfied by F. Thus let u and v be in R^2,

$$u = \begin{bmatrix}u_1\\u_2\end{bmatrix}, v = \begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Let's verify $F(u+v) = F(u) + F(v)$ first. The left side can be expanded into,

$$F(u+v) = F\left(\begin{bmatrix}u_1\\u_2\end{bmatrix} + \begin{bmatrix}v_1\\v_2\end{bmatrix}\right)$$

$$= F\left(\begin{bmatrix}u_1 + v_1\\u_2 + v_2\end{bmatrix}\right)$$

$$= \begin{bmatrix}u_2 + v_2\\u_1 + v_1\end{bmatrix}.$$

The right side can be expanded into,

$$F(u) + F(v) = \begin{bmatrix}u_2\\u_1\end{bmatrix} + \begin{bmatrix}v_2\\v_1\end{bmatrix}$$

$$= \begin{bmatrix}u_2 + v_2\\u_1 + v_1\end{bmatrix}.$$

So we've shown that $F(u+v) = F(u) + F(v)$ holds, since the left side equals to the right side. Similarly, we can verify the $F(cu) = cF(u)$, The left hand side can be expanded into,

$$F(cu) = F\left(c\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) = F\left(\begin{bmatrix}cu_1\\cu_2\end{bmatrix}\right) = \begin{bmatrix}cu_2\\cu_1\end{bmatrix}.$$

The right hand side can be expanded into,

$$\mathrm{cF(u)} = \mathrm{cF}\left(\begin{bmatrix}\mathrm{u}_1\\\mathrm{u}_2\end{bmatrix}\right) = \mathrm{c}\begin{bmatrix}u_2\\u_1\end{bmatrix} = \begin{bmatrix}cu_2\\cu_1\end{bmatrix}.$$

So we've shown that $F(cu) = cF(u)$ holds, since the left side equals to the right side. Therefore function F is a linear transformation.

10. $F: R^2 \to R^2$ defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 1 \end{bmatrix}$$

Answer: We need to determine whether the two linearity properties are satisfied by F. Thus let u and v be in R^2,

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Let's verify $F(u+v) = F(u) + F(v)$ first. The left side can be expanded into,

$$F(u+v) = F\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

$$= F\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} u_1 + v_1 + u_2 + v_2 \\ 1 \end{bmatrix}.$$

The right side can be expanded into,

$$F(u) + F(v) = \begin{bmatrix} u_1 + u_2 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 + v_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + v_1 + u_2 + v_2 \\ 2 \end{bmatrix}.$$

So we've shown that $F(u+v) \neq F(u) + F(v)$ holds, since the left side does not equals to the right side. Therefore function F is not a linear transformation.

11. $F: R^2 \to R^2$ defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \end{bmatrix}$$

Answer: We need to determine whether the two linearity properties are satisfied by F. Thus let u and v be in R^2,

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Let's verify $F(u+v) = F(u) + F(v)$ first. The left side can be expanded into,

$$F(u+v) = F\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)$$

$$= F\left(\begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix}\right)$$

$$= \begin{bmatrix}(u_1+v_1)^2\\(u_1+v_1)(u_2+v_2)\end{bmatrix}.$$

The right side can be expanded into,

$$F(u)+F(v) = \begin{bmatrix}u_1^2\\u_1u_2\end{bmatrix}+\begin{bmatrix}v_1^2\\v_1v_2\end{bmatrix}$$

$$= \begin{bmatrix}u_1^2+v_1^2\\u_1u_2+v_1v_2\end{bmatrix}.$$

So we've shown that $F(u+v) \neq F(u)+F(v)$ holds, since the left side does not equals to the right side. Therefore function F is not a linear transformation.

Ex 26, 30: A linear transformation T is given. In each case find a matrix A such that $T(x) = Ax$. Also describe the null space and the range of T and give the rank and the nullity of T.

26. $T: R^2 \to R^3$ defined by

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1-x_2\\x_1+x_2\\x_2\end{bmatrix}$$

Answer: Look for a matrix A such that $T(x) = Ax$. Since Ax result in a vector of length 3, and x is a vector of length 2, A should be a 3×2 matrix.

$$T(x) = Ax = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1-x_2\\x_1+x_2\\x_2\end{bmatrix}$$

$$\Rightarrow ax_1+bx_2 = x_1-x_2$$

$$cx_1+dx_2 = x_1+x_2$$

$$ex_1+fx_2 = x_2$$

$$\Rightarrow a=1, b=-1, c=1, d=1, e=0, f=1$$

$$\Rightarrow A = \begin{bmatrix}1 & -1\\1 & 1\\0 & 1\end{bmatrix}.$$

By definition, the null space of T, where T is a linear transformation that projects subspace $V \to W$, is $\mathcal{N}(T) = \{v: v \text{ is in } V \text{ and } T(v) = \theta\}$. We've shown that $T(x) = Ax$, the null space of T is also the null space of matrix A, since it contains all x, so that $Ax = \theta$. Therefore, we look for the null space of A,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \to R_2 - R_1 \to \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\to R_3 \leftrightarrow R_2 \to \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\to R_1 + R_2 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\to R_3 - 2R_2 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

There's no free variable in the system, the null space is therefore,

$$\mathcal{N}(T) = \{v \in R^2: v = \theta\}, \dim(\mathcal{N}(T)) = 0$$

Similarly, the range of T is the range of A. As we've shown the two column vectors of A are linearly independent, the range is therefore,

$$\mathcal{R}(T) = \left\{ v \in R^3: v = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \text{where } a, b \text{ are any real numbers} \right\}$$

$$\Rightarrow \dim(\mathcal{R}(T)) = 2$$

30. $T: R^3 \to R$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 2x_1 - x_2 + 4x_3$$

Answer: Look for a matrix A such that $T(x) = Ax$. Since Ax result in a vector of length 1, and x is a vector of length 3, A should be a 1×3 matrix.

$$T(x) = Ax = [a \quad b \quad c]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 - x_2 + 4x_3$$

$$\Rightarrow ax_1 + bx_2 + cx_3 = 2x_1 - x_2 + 4x_3$$

$$\Rightarrow a = 2, b = -1, c = 4$$

$$\Rightarrow A = [2 \quad -1 \quad 4].$$

By definition, the null space of T, where T is a linear transformation that projects subspace $V \to W$, is $\mathcal{N}(T) = \{v: v \text{ is in } V \text{ and } T(v) = \theta\}$. We've shown that $T(x) = Ax$, the null space of T is also the null space of matrix A, since it contains all x, so that $Ax = \theta$. Therefore, we look for the null space of A,

$$A = [2 \quad -1 \quad 4] \to \frac{1}{2}R_1 \to \left[1 \quad -\frac{1}{2} \quad 2\right].$$

There's two free variables in the system, the null space is therefore,

$$\mathcal{N}(T) = \left\{v \in R^3 : v = a\begin{bmatrix}1/2\\1\\0\end{bmatrix} + b\begin{bmatrix}-2\\0\\1\end{bmatrix}, \text{where } a, b \text{ are any real numbers}\right\}$$

$$\Rightarrow \dim\big(\mathcal{N}(T)\big) = 2\,.$$

Similarly, the range of T is the range of A. As we've shown the three column vectors of A are not linearly independent, the range is therefore,

$$\mathcal{R}(T) = \{v \in R^1 : v = a[2], \text{where } a \text{ is any real number}\}$$

$$\Rightarrow \mathcal{R}(T) = R^1, \dim\big(\mathcal{R}(T)\big) = 1\,.$$