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Math 311

Homework #8

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Section 4.1

Ex 1, 4: Find the eigenvalues and the eigenvectors for the given matrix.

1. $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

Answer: Following the procedure of looking for the determinate of matrix $\lambda I - A$.

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 0 \\ -2 & \lambda - 3 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - 4\lambda + 3$$

$$\Rightarrow \lambda = 3 \text{ or } \lambda = 1 .$$

When $\lambda = 3$, plugging in the value of λ, we have

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 3 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ 2x_1 + 3x_2 \end{bmatrix} - 3 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2x_1 \\ 2x_1 \end{bmatrix} = 0$$

$$\Rightarrow x = a \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{where } a \neq 0 .$$

When $\lambda = 1$, plugging in the value of λ, we have

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ 2x_1 + 3x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 \\ 2x_1 + 2x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{where } a \neq 0 .$$

In summary, $\lambda = 1, x = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}, a \neq 0; \lambda = 3, x = a\begin{bmatrix} 0 \\ 1 \end{bmatrix}, a \neq 0;$

2. $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

Answer: Following the procedure of looking for the determinate of matrix $\lambda I - A$.

$$\lambda I - A = \begin{bmatrix} \lambda - 2 & -1 \\ 0 & \lambda + 1 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - \lambda - 2$$

$$\Rightarrow \lambda = 2 \text{ or } \lambda = -1.$$

When $\lambda = 2$, plugging in the value of λ, we have

$$\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ -x_2 \end{bmatrix} - 2 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{where } a \neq 0\,.$$

When $\lambda = -1$, plugging in the value of λ, we have

$$\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ -x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3x_1 + x_2 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \text{where } a \neq 0\,.$$

In summary, $\lambda = 2, x = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0; \lambda = -1, x = a\begin{bmatrix} 1 \\ -3 \end{bmatrix}, a \neq 0;$

4. $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

Answer: Following the procedure of looking for the determinate of matrix $\lambda I - A$.

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 2 \\ -1 & \lambda - 4 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - 5\lambda + 6$$

$$\Rightarrow \lambda = 2 \text{ or } \lambda = 3\,.$$

When $\lambda = 2$, plugging in the value of λ, we have

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 - 2x_2 \\ x_1 + 4x_2 \end{bmatrix} - 2 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + 2x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \text{where } a \neq 0\,.$$

When $\lambda = 3$, plugging in the value of λ, we have

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 3 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 - 2x_2 \\ x_1 + 4x_2 \end{bmatrix} - 3 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2x_1 - 2x_2 \\ x_1 + x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{where } a \neq 0\,.$$

In summary, $\lambda = 2, x = a\begin{bmatrix} 2 \\ -1 \end{bmatrix}, a \neq 0; \lambda = 3, x = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}, a \neq 0;$

Ex 17: Consider the (2×2) symmetric matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Show that there are always real scalar λ such that $A - \lambda I$ is singular.

Answer: Following the procedure of looking for the determinate of matrix $\lambda I - A$, we have

$$\lambda I - A = \begin{bmatrix} \lambda - a & -b \\ -b & \lambda - d \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = (\lambda - a) \times (\lambda - d) - b^2$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - (a + d)\lambda + ad - b^2$$

The discriminate of this quadratic equation $\lambda^2 - (a + d)\lambda + ad - b^2 = 0$ is:

$$(a + d)^2 - 4 \times (ad - b^2)$$

$$= a^2 + d^2 - 2ad + 4b^2$$

$$= (a - b)^2 + 4b^2$$

$$\geq 0$$

So we've shown that the discriminate of this quadratic equation is always greater than or equals to zero, therefore, there will always be solutions for λ. Therefore, we can always find λ so that $\det(\lambda I - A) = 0$, in which case $A - \lambda I$ is singular.

Ex 18: Consider the (2×2) symmetric matrix given by

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \qquad \mathrm{b} \neq 0$$

Show that there are no real scalars λ such that such that $A - \lambda I$ is singular.

Answer: Following the procedure of looking for the determinate of matrix $\lambda I - A$, we have

$$\lambda I - A = \begin{bmatrix} \lambda - a & -b \\ b & \lambda - a \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = (\lambda - a)^2 + b^2$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - 2a\lambda + a^2 + b^2$$

The discriminate of this quadratic equation $\lambda^2 - 2a\lambda + a^2 + b^2 = 0$ is:

$$4a^2 - 4(a^2 + b^2)$$

$$= -4b^2$$

$$< 0, \text{since b} \neq 0$$

So we've shown that the discriminate of this quadratic equation is always less than zero, therefore, there will always be no solution for λ. Hence, we will not be able to find any λ so that $\det(\lambda I - A) = 0$, in which case $A - \lambda I$ is singular.

Ex 19: Let A be a (2×2) matrix. Show that A and A^T have the same set of eigenvalues by considering the polynomial equation (5), $t^2 - (a+d)t + (ad - bc) = 0$.

Answer: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, following the procedure of looking for the determinate of matrix $\lambda I - A$, we have

$$\lambda I - A = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = (\lambda - a)(\lambda - d) - bc$$

We also have $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, following the procedure of looking for the determinate of matrix $\lambda I - A^T$, we have

$$\lambda I - A^T = \begin{bmatrix} \lambda - a & -c \\ -b & \lambda - d \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = (\lambda - a)(\lambda - d) - b^2$$

So, we've shown that the quadratic equations to look for the scalar value λ for matrix A and A^T are the same. Hence, the λ value/values found would be the same. Therefore, we've shown that A and A^T have the same set of eigenvalues.