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Math 311

Homework #9

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Section 4.2

Exercise 12-18: Calculate the determinant of the given matrix. Use Theorem 3 to state whether the matrix is singular or nonsingular.

12. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{bmatrix}$

Answer: Apply row expansion on the first row of the given matrix.

$$\det\left(\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{bmatrix}\right) = 1 \times \det\left(\begin{bmatrix} 3 & 7 \\ 2 & 10 \end{bmatrix}\right) - 2 \times \det\left(\begin{bmatrix} 2 & 7 \\ 4 & 10 \end{bmatrix}\right) + 4 \times \det\left(\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}\right)$$

$$= 1 \times (30 - 14) - 2 \times (20 - 28) + 4 \times (4 - 12)$$

$$= 16 + 16 - 32$$

$$= 0\,.$$

The determinant of the given matrix is 0. According the Theorem 3, $(n \times n)$ matrix A is singular if and only if $\det(A) = 0$, the given matrix is singular.

14. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

Answer: Apply row expansion on the first row of the given matrix.

$$\det\left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}\right) = 1 \times \det\left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\right) - 2 \times \det\left(\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}\right) + 1 \times \det\left(\begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix}\right)$$

$$= 1 \times (3 - 2) - 2 \times (0 + 2) + 1 \times (0 + 3)$$

$$= 1 - 4 + 3$$

$$= 0\,.$$

The determinant of the given matrix is 0. According the Theorem 3, $(n \times n)$ matrix A is singular if and only if $\det(A) = 0$, the given matrix is singular.

18. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer: Apply row expansion on the first row of the given matrix.

$$\det\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0 \times \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) - 1 \times \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$+0 \times \det\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) - 0 \times \det\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= 0 - 1 \times \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) + 0 - 0$$

$$= -\left[0 \times \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) - 1 \times \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + 0 \times \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]$$

$$= -[0 - 1 \times (1 - 0) + 0]$$

$$= 1\,.$$

The determinant of the given matrix is 0. According the Theorem 3, $(n \times n)$ matrix A is singular if and only if $\det(A) = 0$, the given matrix is nonsingular.

Exercise 24: Let A and B be $(n \times n)$ matrices. Use Theorems 2 and 3 to give a quick proof of each of the following.

a. If either A or B is singular, then AB is singular.

Answer: Theorem 2 states, if A and B are $(n \times n)$ matrices, then $\det(AB) = \det(A)\det(B)$. Theorem 3 states, $(n \times n)$ matrix A is singular if and only if $\det(A) = 0$. Therefore, we have,

$$A \text{ is singular} \Rightarrow \det(A) = 0$$

$$\Rightarrow \det(A)\det(B) = 0\det(B) = 0$$

$$\Rightarrow \det(AB) = 0$$

$$\Rightarrow AB \text{ is singular}\,.$$

b. If AB is singular, then either A or B is singular.

Answer: Theorem 2 states, if A and B are $(n \times n)$ matrices, then $\det(AB) = \det(A)\det(B)$. Theorem 3 states, $(n \times n)$ matrix A is singular if and only if $\det(A) = 0$. Therefore, we have,

$$AB \text{ is singular} \Rightarrow \det(AB) = 0$$

$$\Rightarrow \det(A)\det(B) = 0\,.$$

Let's assume that neither A nor B is singular. Then we have,

$$\det(A) \neq 0, \det(B) \neq 0$$

$$\Rightarrow \det(A)\det(B) \neq 0\,.$$

Clearly, our assumption was wrong, therefore, either A or B is singular or both.

Exercise 25: Suppose that A is a $n \times n$ nonsingular matrix, and recall that $\det(I) = 1$, where I is the $(n \times n)$ identity matrix. Show that $\det(A^{-1}) = 1/\det(A)$.

Answer: Theorem 2 states, if A and B are $(n \times n)$ matrices, then $\det(AB) = \det(A)\det(B)$. Theorem 3 states, $(n \times n)$ matrix A is singular if and only if $\det(A) = 0$. A is a $n \times n$ nonsingular matrix, so we have,

$$\because A \text{ is nonsingular}$$

$$\therefore AA^{-1} = I$$

$$\because \det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\therefore \det(I) = \det(A)\det(A^{-1})$$

$$\because \det(I) = 1$$

$$\therefore \det(A)\det(A^{-1}) = 1$$

$$\because A \text{ is nonsingular}$$

$$\therefore \det(A) \neq 0$$

$$\therefore \det(A^{-1}) = \frac{1}{\det(A)}\,.$$

Exercise 26: If A and B are $(n \times n)$ matrices, then usually $AB \neq BA$. Nonetheless, argue that always $\det(AB) = det(BA)$.

Answer: Theorem 2 states, if A and B are $(n \times n)$ matrices, then $\det(AB) = \det(A)\det(B)$. A, B are $n \times n$ nonsingular matrices, so we have,

$$\begin{aligned}\det(AB) &= \det(A)\det(B)\\ &= \det(B)\det(A)\\ &= \det(BA)\,.\end{aligned}$$

Exercise 27: Use Theorem 2 and the conclusion that if A is a $n \times n$ nonsingular matrix $\det(A^{-1}) = 1/\det(A)$ to evaluate the given determinant, where A and B are $(n \times n)$ matrices with $\det(A) = 3$ and $\det(B) = 5$.

$$\det(ABA^{-1})$$

Answer: Theorem 2 states, if A and B are $(n \times n)$ matrices, then $\det(AB) = \det(A)\det(B)$. A, B are $n \times n$ nonsingular matrices, so we have,

$$\begin{aligned}\det(ABA^{-1}) &= \det(A)\det(B)\det(A^{-1})\\ &= \det(A)\det(A^{-1})\det(B)\\ &= 1 \times \det(B)\\ &= 5\,.\end{aligned}$$