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Math 311

Homework #10

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Section 4.3

Exercise 2: Evaluate det (A) by using row operations to introduce zeros into the second and third entries of the first column.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Answer: Apply row operations to introduce zeros to the specified positions.

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \leftrightarrow R_2 - \frac{3}{2}R_1 \leftrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 0 & -5 & -7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\leftrightarrow R_3 - \frac{1}{2}R_1 \leftrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 0 & -5 & -7 \\ 0 & 0 & -2 \end{bmatrix}.$$

According to Theorem 8 that a matrix obtained by adding a fraction of one row to another row has the same determinate as the original matrix, we know that operations, $R_2 - 3/2R_1$ and $R_3 - 1/2R_1$ does not result in a matrix with different determinant. Therefore, we have,

$$\det\left(\begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 4 & 6 \\ 0 & -5 & -7 \\ 0 & 0 & -2 \end{bmatrix}\right).$$

Then apply column expansion on the first column of the matrix above to obtain the determinate.

$$\det\left(\begin{bmatrix} 2 & 4 & 6 \\ 0 & -5 & -7 \\ 0 & 0 & -2 \end{bmatrix}\right) = 2 \times \det\left(\begin{bmatrix} -5 & -7 \\ 0 & -2 \end{bmatrix}\right) - 0 \times \det\left(\begin{bmatrix} 4 & 6 \\ 0 & -2 \end{bmatrix}\right) + 0 \times \det\left(\begin{bmatrix} 4 & 6 \\ -5 & -7 \end{bmatrix}\right)$$

$$= 2 \times \det\left(\begin{bmatrix} -5 & -7 \\ 0 & -2 \end{bmatrix}\right)$$

$$= 2 \times (10 - 0)$$

$$= 20\,.$$

The determinate the given matrix is 20.

Exercise 27: Find examples of (2×2) matrix A and B such thatdet$(A + B) \neq \det(A) + \det(B)$.

Answer: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. To have $\det(A + B) \neq \det(A) + \det(B)$ on (2×2) matrices, we have the following the calculations,

$$\det(A + B) = \det\left(\begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}\right)$$

$$= (a + e)(d + h) - (b + f)(c + g)$$

$$= ad + ah + ed + eh - bc - bg - fc - fg\,.$$

$$\det(A) + \det(B) = ad - bc + eh - fg\,.$$

$$\det(A + B) \neq \det(A) + \det(B)$$

$$\Rightarrow ad + ah + ed + eh - bc - bg - fc - fg \neq ad - bc + eh - fg$$

$$\Rightarrow ah + ed - bg - fc \neq 0\,.$$

Therefore, any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, whose entries satisfy the relation $ah + ed - bg - fc \neq 0$, satisfy the inequality $\det(A + B) \neq \det(A) + \det(B)$. For example the following combination,

$$E.g., A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can verify this example by comparing their determinates. $ah + ed - bg - fc = 1 + 1 - 0 - 0 = 2 \neq 0$.

$$\det(A + B) = \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)$$

$$= 4\,.$$

$$\det(A) + \det(B) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= 1 + 1$$

$$= 2\,.$$

Exercise 28: An $(n \times n)$ matrix A is called *skew symmetric* if $A^T = -A$. Show that if A is skew symmetric then $\det(A) = (-1)^n \det(A)$.

Answer: According the Theorem 5, we have $\det(A^T) = \det(A)$. Therefore we have the following calculations if A is skew symmetric

$$\because \det(A^T) = \det(A) \text{ and } A^T = -A$$

$$\therefore \det(-A) = \det(A)$$

We also know that for a $(n \times n)$ matrix$A$, $\det(cA) = c^n \det(A)$, according to Theorem 7-3.

$$\Rightarrow \det(-A) = (-1)^n \det(A)$$

$$\Rightarrow (-1)^n \det(A) = \det(A)\ .$$

Section 4.4

Exercise 8: Find the characteristic polynomial and the eigenvalues for the given matrix. Also, give the algebraic multiplicity of each eigenvalues.

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix}$$

Answer: We follow the procedures of looking for eigenvalues of the matrix,

$$\lambda I - A = \begin{bmatrix} \lambda+2 & 1 & 0 \\ 0 & \lambda-1 & -1 \\ 2 & 2 & \lambda+1 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+2 & 1 & 0 \\ 0 & \lambda-1 & -1 \\ 2 & 2 & \lambda+1 \end{bmatrix}\right)$$

Apply row operations to simplify the matrix and obtain the characteristic polynomial.

$$\begin{bmatrix} \lambda+2 & 1 & 0 \\ 0 & \lambda-1 & -1 \\ 2 & 2 & \lambda+1 \end{bmatrix} \leftrightarrow R_3 - \frac{2}{\lambda+2}R_1 \leftrightarrow \begin{bmatrix} \lambda+2 & 1 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 2-\dfrac{2}{\lambda+2} & \lambda+1 \end{bmatrix}$$

According the Theorem 7-2, the row operation $R_3 - \frac{2}{\lambda+2}R_1$ does not result in a matrix with different determinate. Therefore,

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+2 & 1 & 0 \\ 0 & \lambda-1 & -1 \\ 2 & 2 & \lambda+1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \lambda+2 & 1 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 2-\dfrac{2}{\lambda+2} & \lambda+1 \end{bmatrix}\right)$$

We apply column expansion to obtain the determinate of the simplified matrix,

$$\det\left(\begin{bmatrix} \lambda+2 & 1 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 2-\dfrac{2}{\lambda+2} & \lambda+1 \end{bmatrix}\right) = (\lambda+2)\times\det\left(\begin{bmatrix} \lambda-1 & -1 \\ 2-\dfrac{2}{\lambda+2} & \lambda+1 \end{bmatrix}\right)$$

$$= (\lambda+2)\times\left[(\lambda-1)(\lambda+1)+\left(2-\frac{2}{\lambda+2}\right)\right]$$

$$= (\lambda+2)(\lambda-1)(\lambda+1)+(2\lambda+2)$$

$$= (\lambda+1)[(\lambda+2)(\lambda-1)+2]$$

$$= (\lambda+1)(\lambda^2-\lambda)$$

$$= \lambda(\lambda+1)(\lambda-1)$$

The characteristic polynomial for the given matrix is $\lambda(\lambda+1)(\lambda-1)$. There are three eigenvalues, $\lambda = 0$ with a algebraic multiplicity of 1, $\lambda = -1$ with a algebraic multiplicity of 1, and $\lambda = 1$ with a algebraic multiplicity of 1.

Exercise 16: Prove property (c) of Theorem 11

Answer: Theorem 11 (c) states, if A is an $(n \times n)$ matrix, and λ is an eigenvalues of A, α is any scalar then $\lambda + \alpha$ is an eigenvalues of $A + \alpha I$.

We have: $Ax = \lambda x$

To be proven: $(A + \alpha I)x = (\lambda + \alpha)x$

$\Rightarrow$ To be proven: $Ax + \alpha Ix = \lambda x + \alpha x$

$\Rightarrow$ To be proven: $\alpha Ix = \alpha x$

$\Rightarrow$ To be proven: $Ix = x$

$Ix = x$ holds for all vector x, therefore, we've proved that $\lambda + \alpha$ is an eigenvalues of $A + \alpha I$.

Exercise 21: An $(n \times n)$ matrix P is called *idempotent* if $P^2 = P$. Show that if P is an invertible idempotent matrix, then $P = I$.

Answer: Theorem 2 states, if A and B are $(n \times n)$ matrices, then $\det(AB) = \det(A)\det(B)$.

$$\because det(AB) = det(A)\, det(B)$$

$$\therefore det(P^2) = det(P)\, det(P) \cdots\cdots\cdots e_1$$

$$\because P^2 = P$$

$$\therefore det(P^2) = det(P) \cdots\cdots\cdots\cdots\cdots\cdots e_2$$

Considering e_1 and e_2, we have $\det(P)\det(P) = \det(P)$.

$$det(P)\, det(P) = det(P)$$

$$\Rightarrow det(P)\,[det(P) - 1] = 0$$

$$\Rightarrow det(P) = 0, 1$$

Since P is invertible, P is not singular, $\det(P) \neq 0$. Therefore $\det(P) = 1$, hence $P = I$.

Exercise 22: Let P be an idempotent matrix. Show that the only eigenvalues of P are $\lambda = 0$ and $\lambda = 1$.

Answer: We've shown in the previous question that the only determinate values for an idempotent matrix are 0 or 1. In other words matrix P is either singular or the identical matrix, I. When $P = I$, we have,

$$Ix = 1 \times x$$

$$\Rightarrow \lambda = 1$$

When P is a singular matrix, then $\lambda = 0$, because to get 0 as an eigenvalue for matrix A, if and only A is a singular matrix. So, we've shown the only possible solutions for the eigenvalues of an idempotent matrix is 0 and 1.

Exercise 20: Find the eigenvalues and eigenvectors for the matrix given

$$\begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}$$

Answer: Follow the procedure of looking for the eigenvalues of the given matrix,

$$A = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} \Rightarrow \lambda I - A = \begin{bmatrix} \lambda - 2 & -4 \\ 2 & \lambda + 2 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = (\lambda - 2)(\lambda + 2) + 8 = \lambda^2 + 4$$

$$\Rightarrow \lambda = \pm 2i$$

Look for the eigenvectors for this matrix of different eigenvalues. When $\lambda = 2i$, we have,

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2i \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + 4x_2 \\ -2x_1 - 2x_2 \end{bmatrix} - \begin{bmatrix} 2ix_1 \\ 2ix_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} (2 - 2i)x_1 + 4x_2 \\ -2x_1 - (2 + 2i)x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 1 + i \\ -1 \end{bmatrix}$$

When $\lambda = -2i$, we have,

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2i \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + 4x_2 \\ -2x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} 2ix_1 \\ 2ix_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} (2 + 2i)x_1 + 4x_2 \\ -2x_1 - (2 - 2i)x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 1 - i \\ -1 \end{bmatrix}$$

Therefore the eigenvalues and eigenvectors for this system are,

$$\lambda = 2i, x = \left\{ x \in R^2 : x = a \begin{bmatrix} 1 + i \\ -1 \end{bmatrix}, \text{where } a \text{ is any real number} \right\}$$

$$\lambda = -2i, x = \left\{ x \in R^2 : x = a \begin{bmatrix} 1 - i \\ -1 \end{bmatrix}, \text{where } a \text{ is any real number} \right\}$$