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Math 311

Homework #10

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Section 4.7

Exercise 2-10: Determine whether the given matrix A is diagonalizable. If A is diagonalizable, calculate A^5.

2. $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Answer: First, we look for the eigenvalues, and use the number of eigenvalues to determine whether the given matrix is diagonalizable.

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - 2\lambda$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda = 2\,.$$

When $\lambda = 0$, plugging in the value of λ, we have

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 0 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{where } a \neq 0\,.$$

When $\lambda = 2$, plugging in the value of λ, we have

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} - 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -x_1 - x_2 \\ -x_1 - x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{where } a \neq 0\,.$$

In summary, $\lambda = 0, x = a\begin{bmatrix}1\\1\end{bmatrix}, a \neq 0; \lambda = 2, x = a\begin{bmatrix}1\\-1\end{bmatrix}, a \neq 0$. The corollary of Theorem 19 states if A is a $(n \times n)$ matrix and has n distinct eigenvalues then A is similar to a diagonal matrix (A is diagonalizable). Since we have two distinct eigenvalues for the given (2×2) matrix A, we've shown A is diagonalizable.

$$\lambda = 0, x = \left\{x \in R^2 : x = a\begin{bmatrix}1\\1\end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0\right\}$$

$$\lambda = 2, x = \left\{x \in R^2 : x = a\begin{bmatrix}1\\-1\end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0\right\}$$

$$\Rightarrow S = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix}0.5 & 0.5\\0.5 & -0.5\end{bmatrix}$$

$$\Rightarrow D = S^{-1}AS = \begin{bmatrix}0.5 & 0.5\\0.5 & -0.5\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}.$$

According to the definition of similar matrices, $D = S^{-1}AS$, and $A = SDS^{-1}$. We also know that $(SDS^{-1})^n = SD^nS^{-1}$. Therefore the calculation below follows,

$$\begin{aligned}
A^5 &= (SDS^{-1})^5\\
&= SD^5S^{-1}\\
&= \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}^5\begin{bmatrix}0.5 & 0.5\\0.5 & -0.5\end{bmatrix}\\
&= \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 32\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0.5 & -0.5\end{bmatrix}\\
&= \begin{bmatrix}0 & 32\\0 & -32\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0.5 & -0.5\end{bmatrix}\\
&= \begin{bmatrix}16 & -16\\-16 & 16\end{bmatrix}.
\end{aligned}$$

4. $A = \begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}$

Answer: First, we look for the eigenvalues, and use the number of eigenvalues to determine whether the given matrix is diagonalizable.

$$\lambda I - A = \begin{bmatrix}\lambda - 1 & -3\\0 & \lambda - 1\end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - 2\lambda + 1$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda = 1, \text{algebraic multiplicity of } \lambda \text{ is } 2.$$

When $\lambda = 1$, plugging in the value of λ, we have

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -3 \\ 0 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix} \to -\frac{1}{3}R_1 \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \mathcal{N}(\lambda I - A) = \left\{ x \in R^2 \colon x = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{where } a \text{ is any real number} \right\}$$

$$\Rightarrow \dim\big(\mathcal{N}(\lambda I - A)\big) = 1 = \text{geometric multipliciy (of } \lambda = 1)$$

$$\Rightarrow \text{geometric multiplicy} < \text{algebraic multiplicity (of } \lambda = 1)$$

The corollary of Theorem 19 states if A is a $(n \times n)$ matrix, A is similar to a diagonal matrix (A is diagonalizable) if and only if the geometric multiplicity of $\lambda_i =$ the algebraic multiplicity of λ_i. Since we have one eigenvalues $\lambda = 1$ for the given matrix and its geometric multiplicity is strictly less than its algebraic multiplicity, we've shown A is not diagonalizable.

6. $A = \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix}$

Answer: First, we look for the eigenvalues, and use the number of eigenvalues to determine whether the given matrix is diagonalizable.

$$\lambda I - A = \begin{bmatrix} \lambda + 1 & -7 \\ 0 & \lambda - 1 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - 1$$

$$\Rightarrow \lambda = 1 \text{ or } \lambda = -1\,.$$

When $\lambda = 1$, plugging in the value of λ, we have

$$\begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -x_1 + 7x_2 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2x_1 + 7x_2 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 7 \\ 2 \end{bmatrix}, \text{where } a \neq 0\,.$$

When $\lambda = -1$, plugging in the value of λ, we have

$$\begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -x_1 + 7x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 7x_2 \\ 2x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{where } a \neq 0 .$$

In summary, $\lambda = 1, x = a \begin{bmatrix} 7 \\ 2 \end{bmatrix}, a \neq 0; \lambda = -1, x = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0$. The corollary of Theorem 19 states if A is a $(n \times n)$ matrix and has n distinct eigenvalues then A is similar to a diagonal matrix (A is diagonalizable). Since we have two distinct eigenvalues for the given (2×2) matrix A, we've shown A is diagonalizable.

$$\lambda = 1, x = \left\{ x \in R^2 : x = a \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\lambda = -1, x = \left\{ x \in R^2 : x = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\Rightarrow S = \begin{bmatrix} 7 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix} 0 & 0.5 \\ 1 & -3.5 \end{bmatrix}$$

$$\Rightarrow D = S^{-1}AS = \begin{bmatrix} 0 & 0.5 \\ 1 & -3.5 \end{bmatrix} \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

According to the definition of similar matrices, $D = S^{-1}AS$, and $A = SDS^{-1}$. We also know that $(SDS^{-1})^n = SD^nS^{-1}$. Therefore the calculation below follows,

$$\begin{aligned} A^5 &= (SDS^{-1})^5 \\ &= SD^5S^{-1} \\ &= \begin{bmatrix} 7 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^5 \begin{bmatrix} 0 & 0.5 \\ 1 & -3.5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0.5 \\ 1 & -3.5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.5 \\ 1 & -3.5 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

10. $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: First, we look for the eigenvalues, and use the number of eigenvalues to determine whether the given matrix is diagonalizable. Notice that this is an upper triangular matrix. We know that for an upper triangular matrix, the characteristic polynomial is simply $\prod_i^n(\lambda - t_{ii})$, where t_{ii} is the corresponding value on the main diagonal. Therefore we can obtain the eigenvalues directly from the matrix. They are, $\lambda = 1$, with a algebraic multiplicity of 2, and $\lambda = 2$, with a algebraic multiplicity of 1.

When $\lambda = 1$, plugging in the value of λ, we have

$$\lambda_1 I - A = \begin{bmatrix} \lambda - 1 & -1 & 1 \\ 0 & \lambda - 2 & 1 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to -R_2 \to \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to R_1 + R_2 \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \mathcal{N}(\lambda_1 I - A) = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{where } a, b \text{ are any real numbers} \right\}$$

$$\Rightarrow \dim\big(\mathcal{N}(\lambda_1 I - A)\big) = 2 = \text{geometric multipliciy (of } \lambda_1 = 1)$$

$$\Rightarrow \text{geometric multiplicy} = \text{algebraic multiplicity (of } \lambda_1 = 1)\,.$$

When $\lambda = 2$, plugging in the value of λ, we have

$$\lambda_2 I - A = \begin{bmatrix} \lambda - 1 & -1 & 1 \\ 0 & \lambda - 2 & 1 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to R_1 - R_3 \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to R_2 - R_3 \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \mathcal{N}(\lambda_2 I - A) = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{where } a \text{ is any real number} \right\}$$

$$\Rightarrow \dim\big(\mathcal{N}(\lambda_2 I - A)\big) = 1 = \text{geometric multipliciy (of } \lambda_2 = 2)$$

$$\Rightarrow \text{geometric multiplicy} = \text{algebraic multiplicity (of } \lambda_2 = 2)\,.$$

The corollary of Theorem 19 states if A is a $(n \times n)$ matrix, A is similar to a diagonal matrix (A is diagonalizable) if and only if the geometric multiplicity of λ_i = the algebraic multiplicity of λ_i. Since we've shown the geometric multiplicities equal to the algebraic multiplicities respectively, we've proved A is diagonalizable. When $\lambda = 1$, plugging in the value of λ to look for eigenvectors, we have

$$\mathcal{N}(\lambda_1 I - A) = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{where } a, b \text{ are any real numbers} \right\}$$

$$\Rightarrow x = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{and } x = b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ are eigenvectors for } \lambda_1 = 1$$

$$\mathcal{N}(\lambda_2 I - A) = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{where } a \text{ is any real number} \right\}$$

$$\Rightarrow x = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{are eigenvectors for } \lambda_2 = 2$$

$$\Rightarrow S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now we can Compute S^{-1}, and further compute D, which could also be obtained by putting the eigenvalues on the main diagonal.

$$S \mid I = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to R_3 - R_2 \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$\to -R_3 \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

$$\to R_1 - R_3 \to \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

$$\to R_2 - R_3 \to \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

$$\Rightarrow D = S^{-1}AS = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

According to the definition of similar matrices, $D = S^{-1}AS$, and $A = SDS^{-1}$. We also know that $(SDS^{-1})^n = SD^nS^{-1}$. Therefore the calculation below follows,

$$
\begin{aligned}
A^5 &= (SDS^{-1})^5 \\
&= SD^5S^{-1} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 32 \\ 0 & 1 & 32 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 31 & -31 \\ 0 & 32 & -31 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$