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Math 412
Homework #3 (partial)
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Question: Find the lattice of ideals of the ring $\mathbb{Z}_{13}$

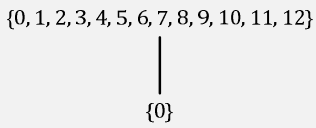
Answer: For an arbitrary ring $(R, +, \cdot)$, let $(R, +)$ be the underlying additive group. A subset l is called an *ideal* of R if $(l, +)$ is a subgroup of $(R, +)$ and for all $x \in l, r \in R, x \cdot r \in l, r \cdot x \in l$. The *sum* and *intersection* of ideals is again an ideal. With these two operations as join and meet, the set of all ideals of a given ring forms a complete modular lattice. The sum operation is defined as below, where l_a and l_b are two ideals.

$$a + b \coloneqq \{a + b \mid a \in l_a \text{ and } b \in l_b\}$$

The ring $\mathbb{Z}_{13}$ has the following multiplication characteristic table, from which we can see that every *principle ideal* generated by a single element, except element 0, contains the whole set. Every row / column, except for 0 element, contains all 13 elements of $\mathbb{Z}_{13}$. This is due to the fact that 13 is a prime number.

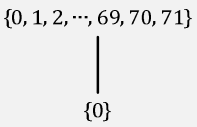
·	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	1	3	5	7	9	11
3	0	3	6	9	12	2	5	8	11	1	4	7	10
4	0	4	8	12	3	7	11	2	6	10	1	5	9
5	0	5	10	2	7	12	4	9	1	6	11	3	8
6	0	6	12	5	11	4	10	3	9	2	8	1	7
7	0	7	1	8	2	9	3	10	4	11	5	12	6
8	0	8	3	11	6	1	9	4	12	7	2	10	5
9	0	9	5	1	10	6	2	11	7	3	12	8	4
10	0	10	7	4	1	11	8	5	2	12	9	6	3
11	0	11	9	7	5	3	1	12	10	8	6	4	2
12	0	12	11	10	9	8	7	6	5	4	3	2	1

Therefore, there are only two ideals in $\mathbb{Z}_{13}$, $\{0\}$ and $\mathbb{Z}_{13}$ itself. The lattice of ideals, hence, contains two elements.



Question: Find the lattice of ideals of the ring $\mathbb{Z}_{71}$

Answer: Similarly, since 71 is a prime number, the lattice of ideal for ring $\mathbb{Z}_{71}$ contains only two elements, $\{0\}$ and $\mathbb{Z}_{71}$ itself.

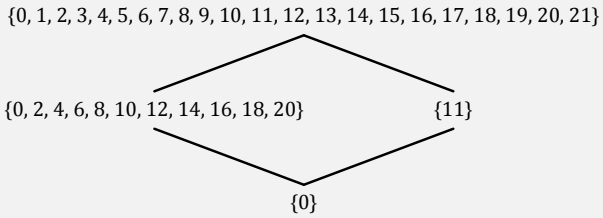


Question: Find the lattice of ideals of the ring $\mathbb{Z}_{22}$

Answer: We will look for the principle ideals by evaluating each element in $\mathbb{Z}_{22}$.

Single element	Principle Ideal
Element 0:	$\{0\}$
Element 1:	$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$
Element 2:	$\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$
:	:
Element 11:	$\{0, 11\}$
:	:
Element 20:	$\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$
Element 21:	$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

There are four ideals in ring $\mathbb{Z}_{22}$. The lattice of the ideals for $\mathbb{Z}_{22}$ contains four elements.

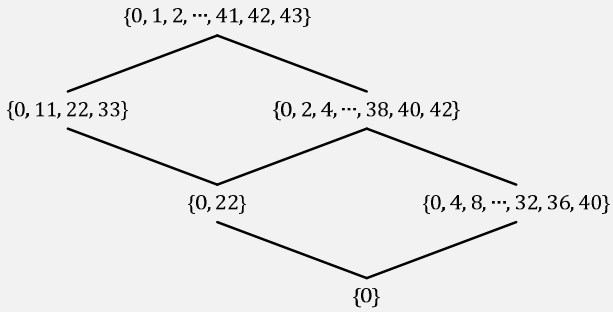


Question: Find the lattice of ideals of the ring $\mathbb{Z}_{44}$

Answer: We will look for the principle ideals by evaluating each element in $\mathbb{Z}_{44}$.

Single element	Principle Ideal
Element 0:	$\{0\}$
All odd elements except multiples of 11:	$\{0, 1, 2, \cdots, 41, 42, 43\}$
All even elements except multiples of 4 or 11:	$\{0, 2, 4, \cdots, 38, 40, 42\}$
All multiples of 4:	$\{0, 4, 8, \cdots, 32, 36, 40\}$
Element 11 and 33:	$\{0, 11, 22, 33\}$
Element 22:	$\{0, 22\}$

There are 7 ideals in ring $\mathbb{Z}_{22}$. The lattice of the ideals for $\mathbb{Z}_{22}$ contains seven elements.

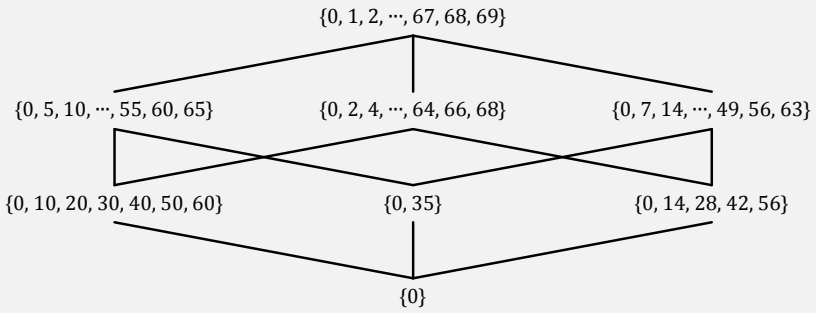


Question: Find the lattice of ideals of the ring $\mathbb{Z}_{70}$

Answer: We will look for the principle ideals by evaluating each element in $\mathbb{Z}_{70}$.

Single element	Principle Ideal
Element 0:	$\{0\}$
All odd elements except multiples of 5 or 7:	$\{0, 1, 2, \cdots, 67, 68, 69\}$
All even elements except multiples of 5 or 7:	$\{0, 2, 4, \cdots, 64, 66, 68\}$
All multiples of 5 but not 2 or 7:	$\{0, 5, 10, \cdots, 55, 60, 65\}$
All multiples of 10:	$\{0, 10, 20, 30, 40, 50, 60\}$
All multiples of 7 but not 2 or 5:	$\{0, 7, 14, \cdots, 49, 56, 63\}$
All multiples of 14:	$\{0, 14, 28, 42, 56\}$
Element 35:	$\{0, 35\}$

There are 8 ideals in ring $\mathbb{Z}_{22}$. The lattice of the ideals for $\mathbb{Z}_{22}$ contains eight elements.



Question: Find the lattice of ideals of the rings $\mathbb{Z}_{140}$

Answer: We will look for the principle ideals by evaluating each element in $\mathbb{Z}_{140}$.

Single element	Principle Ideal
Element 0:	$\{0\}$
All odd elements except multiples of 5 or 7:	$\{0, 1, 2, \cdots, 137, 138, 139\}$
All even elements except multiples of 4, 5 or 7:	$\{0, 2, 4, \cdots, 134, 136, 138\}$
All multiples of 4 but not 5 or 7:	$\{0, 4, 8, \cdots, 128, 132, 136\}$
All multiples of 5 but not 2 or 7:	$\{0, 5, 10, \cdots, 125, 130, 135\}$
All multiples of 7 but not 2 or 5:	$\{0, 7, 14, \cdots, 119, 126, 133\}$
All multiples of 10 but not 4 or 7:	$\{0, 10, 20, \cdots, 110, 120, 130\}$
All multiples of 14 but not 4:	$\{0, 14, 28, \cdots, 98, 112, 126\}$
All multiples of 20:	$\{0, 20, 40, 60, 80, 100, 120\}$
All multiples of 28:	$\{0, 28, 56, 84, 112\}$
All multiples of 35 but not 70:	$\{0, 35, 70, 105\}$
All multiples of 70:	$\{0, 70\}$

There are 12 ideals in ring $\mathbb{Z}_{22}$. The lattice of the ideals for $\mathbb{Z}_{22}$ contains twelve elements.