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Math 412

Homework #3 (Textbook)

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Exercise 4: If A and B are ideals of R, prove $A \cap B$ is an ideal of R.

Answer: For an arbitrary ring $(R, +, \cdot)$, let $(R, +)$ be the underlying additive group. A subset I is called an ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and for all $x \in I$ and for all $r \in R$, $r \cdot x = x \cdot r \in I$.

It is clear that $A \cap B$ is a subgroup of $(R, +)$. It is closed under addition.

$$a, b \in A \cap B$$

$$\Rightarrow a, b \in A, \text{and } a, b \in B$$

$$\Rightarrow a + b \in A \text{ and } a + b \in B$$

$$\Rightarrow a + b \in A \cap B\,.$$

The addition operation is associative, since $A \cap B$ is a subset of R. The additive identity is 0_R, since $0_R \in A$, and $0_R \in B$. The inverses of element x in $A \cap B$ is $-x$, since $x, -x \in A$, $x, -x \in B$.

Now we need to show $r \cdot x = x \cdot r \in A \cap B$ for all $r \in R$ and $x \in A \cap B$.

$$x \in A \cap B$$

$$\Rightarrow x \in A \text{ and } x \in B$$

$$A, B \text{ are ideals of } R$$

$$\Rightarrow r \cdot x = x \cdot r \in A \text{ and } r \cdot x = x \cdot r \in B$$

$$\Rightarrow r \cdot x = x \cdot r \in A \cap B\,.$$

At this point, we've shown that $(A \cap B, +)$ is a subgroup of $(R, +)$ and for all $x \in A \cap B$ and $r \in R$, $r \cdot x = x \cdot r \in A \cap B$. $A \cap B$ is, therefore, an ideal of ring R.

Exercise 5: If A and B are ideals of R, prove their sum, defined by

$$A + B = \{a + b : a \in A, b \in B\},$$

is an ideal of R and is the smallest ideal that contains both A and B.

Answer: We will first show that $A + B$ is an *ideal* of R. $A + B$ is closed under addition. Let's have $c_1, c_2 \in A + B$, and $c_1 = a_1 + b_1, c_2 = a_2 + b_2$, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

$$a_1, a_2 \in A \text{ and } b_1, b_2 \in B$$

$$c_1 = a_1 + b_1 \text{ and } c_2 = a_2 + b_2$$

$$\Rightarrow c_1 + c_2 = a_1 + b_1 + a_2 + b_2 = (a_1 + a_2) + (b_1 + b_2) \in A + B\,.$$

The addition operation is associative, since $A + B$ is a subset of R. The additive identity is 0_R, since $0_R \in A$, $0_R \in B$, $0_R = 0_R + 0_R \in A + B$. The inverse of $a + b$ in $A + B$ where $a \in A$, $b \in B$, is $-a - b$, since $-a \in A$, $-b \in B$.

Now we need to show $r \cdot x = x \cdot r \in A + B$ for all $r \in R$ and $x \in A + B$.

$$x \in A + B$$

$$\Rightarrow x = a + b, \text{where } a \in A, b \in B$$

$$\Rightarrow r \cdot x = r \cdot (a + b) = r \cdot a + r \cdot b$$

$$\Rightarrow x \cdot r = (a + b) \cdot r = a \cdot r + b \cdot r$$

$$r \cdot a = a \cdot r \in A \; and \; r \cdot b = b \cdot r \in B$$

$$\Rightarrow r \cdot x = x \cdot r \in A + B\,.$$

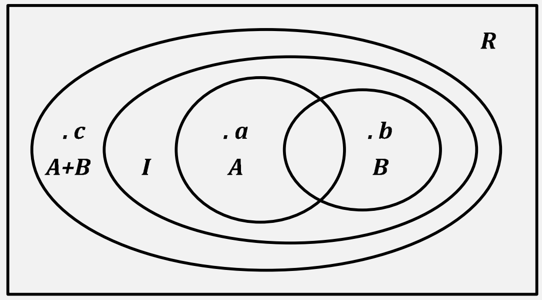
At this point, we've shown that $(A + B, +)$ is a subgroup of $(R, +)$ and for all $x \in A + B$ and $r \in R$, $r \cdot x = x \cdot r \in A + B$. $A + B$ is, therefore, an ideal of ring R.

Now we will show $A \subseteq A + B$ and $B \subseteq A + B$. Without losing generosity, we need only to prove one. We will show A is *contained* in $A + B$. Let's have $a \in A$.

$$a \in A, 0_R \in B$$

$$\Rightarrow a = a + 0_R \in A + B\,.$$

We've shown any element in A is also an element in $A + B$. A is, therefore, contained in $A + B$. We also need to show $A + B$ is the *smallest* ideal that contains both A and B. Let assume there is another ideal, I, that's smaller than $A + B$, $I \subset A + B$, and contains A and B, $A, B \subseteq I$. We now pick an element, c, from $A + B$, but not contained in I. $c = a + b \in A + B, c \notin I$.



Now we have a contradictory.

$$a \in A, b \in B$$

$$\Rightarrow a, b \in I$$

$$\Rightarrow c = a + b \in I\ .$$

The assumption, $c \notin I$, is therefore not true. There is no such element, c, that $c \in A + B$, but $c \notin I$. The set I is therefore the same as $A + B$. $A + B$ is therefore the smallest ideal that contains both A and B.

Exercise 11: Let R be a commutative ring with identity and $x \in R$. Give a complete proof that the set (x) defined by $(x) = \{rx : r \in R\}$ is an ideal of R.

Answer: We will first show that (x) is a subgroup of $(R, +)$. (x) is closed under addition. Let's have $c_1, c_2 \in (x)$, and $c_1 = r_1 x, c_2 = r_2 x$, where $r_1, r_2 \in R$.

$$c_1 = r_1 x \text{ and } c_2 = r_2 x$$

$$\Rightarrow c_1 + c_2 = r_1 x + r_2 x = (r_1 + r_2)x \text{ since } r_1, r_2, x \in R$$

$$\Rightarrow c_1 + c_2 = (r_1 + r_2)x \in (x)\ .$$

The addition operation is associative, since (x) is a subset of R. The additive identity is 0_R, since $0_R \in R$, $0_R x = 0_R \in (x)$. The inverse of rx in (x) is $(-r)x = -rx$.

Now we need to show $r \cdot c = c \cdot r \in (x)$ for all $r \in R$ and $c \in (x)$, where $c = r'x, r' \in R$.

$$R \text{ is commutative}$$

$$\Rightarrow r \cdot c = c \cdot r$$

$$c = r'x \in (x)$$

$$\Rightarrow r \cdot c = c \cdot r = r \cdot r' \cdot x = (r \cdot r') \cdot x \in (x)\ .$$

At this point, we've shown that $((x), +)$ is a subgroup of $(R, +)$ and for all $c \in (x)$ and $r \in R$, $r \cdot c = c \cdot r \in (x)$. (x) is, therefore, an ideal of ring R.

Exercise 12: Let R be a commutative ring with identity such that the only ideals of R are the two trivial ideals (0) and R. Prove that every nonzero element $x \in R$ has a multiplicative inverse.

Answer: R has only the two trivial ideals. In other words, a principle ideal generated by any nonzero element, a, contains all elements in R including the multiplicative identity, 1_R. In order to generate 1_R, there must exist an element b, such that $a \cdot b = b \cdot a = 1_R$. Hence, we've shown that for any element, a, in R, it has a multiplicative inverse, b.