Student: Yu Cheng (Jade)

Math 412

Homework #4

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Homework #4—Page 166

Exercise 4: Prove that $(ab)^2 = a^2b^2$ for all choices of a, b in group G if and only if G is Abelian.

Answer: We will first show $(ab)^2 = a^2b^2$ for any $a, b \in G$ derives that G is an Abelian group, the *if* half.

$$abab = aabb$$

$$\Rightarrow a^{-1}ababb^{-1} = a^{-1}aabbb^{-1}$$

$$\Rightarrow ba = ab\,.$$

If $(ab)^2 = a^2b^2$ for any $a, b \in G$, it follows that G is an Abelian group. Now we need to show the *only if* half. If G is Abelian group, we can just reverse the procedure above and derive $(ab)^2 = a^2b^2$. Therefore, $(ab)^2 = a^2b^2$ is a sufficient and necessary condition for G being an Abelian group.

Exercise 7: Let a be a fixed element of a group G. Prove that the set

$$C_G(a) = \{x \in G : ax = xa\}$$

is a subgroup of G. $C_G(a)$ is called the *centralizer* of a in G.

Answer: First, we will show that $C_G(a)$ is closed under group G operation, $\cdot$.

$$x_1 \in C_G(a), \text{and } x_2 \in C_G(a)$$

$$\Rightarrow a \cdot x_1 = x_1 \cdot a, \text{and } a \cdot x_2 = x_2 \cdot a$$

$$\Rightarrow (x_1 \cdot x_2) \cdot a = x_1 \cdot x_2 \cdot a = x_1 \cdot a \cdot x_2 = a \cdot x_1 \cdot x_2 = a \cdot (x_1 \cdot x_2)$$

$$\Rightarrow x_1 \cdot x_2 \in C_G(a)\,.$$

The group operation, $\cdot$, is associative, since $C_G(a)$ is a subset of G. The identity, $e \in C_G(a)$.

$$a \cdot e = e \cdot a = a$$

$$\Rightarrow e \in C_G(a)\,.$$

The inverse of $x \in C_G(a)$ is x^{-1}, the same as its inverse in G.

$$ax = xa$$

$$\Rightarrow x^{-1}axx^{-1} = x^{-1}xax^{-1}$$

$$\Rightarrow x^{-1}a = ax^{-1}$$

$$\Rightarrow x^{-1} \in C_G(a)\,.$$

At this point we've shown that $C_G(a)$ satisfies all conditions for a group, and $C_G(a)$ is a subset of G. $C_G(a)$ is, therefore, a subgroup of G.

Exercise 8: For any group G the set $Z(G) = \{b \in G : bc = cb \text{ for all } c \in G\}$ is called the center of G. Prove $Z(G)$ is a subgroup of G. Show also that $Z(G) = G$ if and only if G is Abelian.

Answer: We will first show that $Z(G)$ is a *subgroup* of G. $Z(G)$ is closed under group G operation, $\cdot$.

$$x_1 \in Z(G), \text{and } x_2 \in Z(G)$$

$$\Rightarrow a_1 \cdot x_1 = x_1 \cdot a_1, \text{and } a_2 \cdot x_2 = x_2 \cdot a_2, \text{where } a_1, a_2, x_1, x_2 \in G$$

$$\Rightarrow a_1 \cdot x_1 \cdot a_2 \cdot x_2 = x_1 \cdot a_1 \cdot x_2 \cdot a_2$$

$$\Rightarrow a_1 \cdot a_2 \cdot x_1 \cdot x_2 = x_1 \cdot x_2 \cdot a_1 \cdot a_2$$

$$\Rightarrow (a_1 \cdot a_2) \cdot (x_1 \cdot x_2) = (x_1 \cdot x_2) \cdot (a_1 \cdot a_2)$$

$$\Rightarrow x_1 \cdot x_2 \in Z(G)\,.$$

The group operation, $\cdot$, is associative, since $Z(G)$ is a subset of G. The identity, $e \in C_G(a)$.

$$a \cdot e = e \cdot a = a \text{ for all } a \in G$$

$$\Rightarrow e \in Z(G)\,.$$

The inverse of $x \in Z(G)$ is x^{-1}, the same as its inverse in G.

$$ax = xa \text{ for all } a \in G$$

$$\Rightarrow x^{-1}axx^{-1} = x^{-1}xax^{-1}$$

$$\Rightarrow x^{-1}a = ax^{-1} \text{ for all } a \in G$$

$$\Rightarrow x^{-1} \in Z(G)\,.$$

At this point we've shown that$Z(G)$ satisfies all conditions for a group, and $Z(G)$ is a subset of G. $Z(G)$ is, therefore, a subgroup of G.

Now we will show that $Z(G) = G$ is the sufficient and necessary condition for G being an *Abelian* group. $Z(G) = G$ means for all pairs of $a, b \in G$, $bc = cb$ holds. This is precisely the definition of Abelian group. If G is an Abelian group, $bc = cb$ holds for all $a, b \in G$, which indicates $Z(G) = G$.

Homework #4—Page 194

Exercise 10: Let H be a subgroup of a group G. Show that there is a well-defined correspondence β from the set of left cosets of H to the set of right cosets of H that satisfies $\beta(aH) = Ha^{-1}$. Use this to show that the number of left cosets of H in G is the same as the number of right cosets of H in G.

Answer: We will first show that β, defined as $\beta(aH) = Ha^{-1}$, is well-defined. Namely, we want to show that β is a *map* from the left cosets to the right cosets, and $\beta(aH) = Ha^{-1}$ holds no matter what element a we choose from G. Let's have $(aH) = Ha^{-1}$, $(bH) = Hb^{-1}$, and let's assume $Ha^{-1} = Hb^{-1}$.

$$Ha^{-1} = Hb^{-1}$$

$$\Rightarrow Ha^{-1}b = Hb^{-1}b$$

$$\Rightarrow Ha^{-1}b = H$$

$$\Rightarrow a^{-1}b \in H$$

$$\Rightarrow (a^{-1}b)^{-1} \in H$$

$$\Rightarrow b^{-1}a \in H$$

$$\Rightarrow b^{-1}aH = H$$

$$\Rightarrow bb^{-1}aH = bH$$

$$\Rightarrow aH = bH\,.$$

We've shown that β is well-defined. There is no ambiguous. The relation holds regardless of what a wee choose. β is also *onto*. For any given Ha^{-1}, aH always exists. So $\beta(aH) = Ha^{-1}$ is a one-to-one and onto map from the left cosets of H in G to the right cosets of H in G. Hence, the number of left cosets of H in G is the same as the number of right cosets of H in G.

Exercise 1: Let a be an element of the group G and let $\phi : G \to G$ be the function defined by $\phi(g) = aga^{-1}$. Prove that ϕ is an isomorphism of G with itself.

Answer: A one-to-one correspondence $\phi : G_1 \to G_2$ between two groups G_1 and G_2 is called a group isomorphism if $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ holds for all $g_1, g_2 \in G_1$.

It is clear that ϕ, defined as $\phi(g) = aga^{-1}$, is a one-to-one map from G to itself. Let's have $g_1, g_2 \in G$, and assume $ag_1a^{-1} = ag_2a^{-1}$.

$$
\begin{aligned}
&ag_1a^{-1} = ag_2a^{-1} \\
&\Rightarrow a^{-1}ag_1a^{-1}a = a^{-1}ag_2a^{-1}a \\
&\Rightarrow g_1 = g_2 \,.
\end{aligned}
$$

Now we will show that $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ holds for all $g_1, g_2 \in G$.

$$
\begin{aligned}
\phi(g_1)\phi(g_2) &= ag_1a^{-1}ag_2a^{-1} \\
&= ag_1g_2a^{-1} \\
&= \phi(g_1g_2)\,.
\end{aligned}
$$

At this point we've shown that $\phi : G \to G$ is an isomorphism. It can be considered as simply renaming the elements of G, since it's an one-to-one and onto function.

Exercise 3: If G is a group, prove that the set $Z(G) = \{a : a \in G \text{ and } ax = xa \text{ for every } x \in G\}$ is a normal subgroup of G. [$Z(G)$ is called the *center* of G.]

Answer: To show that $Z(G)$ is a normal subgroup of G, we need to first show that $Z(G)$ is a subgroup of G, which we've already done in Exercise 8 of page 166. Then we need to show that it is closed under conjugation over G, $gag^{-1} \in Z(G)$, for any $g \in G$, and $a \in Z(G)$.

$$
\begin{aligned}
&a \in Z(G) \Rightarrow ga = ag \ \text{ for all } g \in G \\
&\Rightarrow gag^{-1} = agg^{-1} = a \\
&\Rightarrow gag^{-1} \in Z(G)\,.
\end{aligned}
$$

Therefore $Z(G)$ is a normal subgroup of G.

Exercise 11: Let G be any group and H a subgroup that contains every element $aba^{-1}b^{-1}$ with $a, b \in G$. Prove that H is a normal subgroup of G and G/H is Abelian.

Answer: We will first show that H is a normal subgroup of G. Let $g \in G, x \in H$, and $x = aba^{-1}b^{-1}$.

$$\begin{aligned} gxg^{-1} &= g \cdot aba^{-1}b^{-1} \cdot g^{-1} \\ &= ga \cdot (g^{-1}g) \cdot b \cdot (g^{-1}g) \cdot a^{-1} \cdot (g^{-1}g) \cdot b^{-1}g^{-1} \\ &= (gag^{-1}) \cdot (gbg^{-1}) \cdot (ga^{-1}g^{-1}) \cdot (gb^{-1}g^{-1}) \\ &= (gag^{-1}) \cdot (gbg^{-1}) \cdot (gag^{-1})^{-1} \cdot (gbg^{-1})^{-1} \\ &\Rightarrow gxg^{-1} \in H\,. \end{aligned}$$

We've shown $gxg^{-1} \in H$ holds for any $g \in G$ and $x \in H$. H is therefore a normal subgroup of G. Now we need to show that G/H is Abelian. Let $aH, bH \in G/H$. Since $a, b \in G$, $aba^{-1}b^{-1} \in H$, so does $bab^{-1}a^{-1} \in H$.

$$aba^{-1}b^{-1} \in H$$

$$\Rightarrow aba^{-1}b^{-1}H = H$$

$$\Rightarrow (aH)(bH)(a^{-1}H)(b^{-1}H) = H$$

$$bab^{-1}a^{-1} \in H$$

$$\Rightarrow bab^{-1}a^{-1}H = H$$

$$\Rightarrow (bH)(aH)(b^{-1}H)(a^{-1}H) = H$$

$$\Rightarrow (aH)(bH)(a^{-1}H)(b^{-1}H) = (bH)(aH)(b^{-1}H)(a^{-1}H)$$

$$\Rightarrow (aH)(bH)(a^{-1}H)(b^{-1}H)(bH)(aH) = (bH)(aH)(b^{-1}H)(a^{-1}H)(bH)(aH)$$

$$\Rightarrow (aH)(bH) = (bH)(aH)(b^{-1}H)(a^{-1}H)(bH)(aH)$$

$$\Rightarrow (aH)(bH) = (bH)(aH)(b^{-1}a^{-1}ba)H$$

$$\Rightarrow (aH)(bH) = (bH)(aH)\,.$$

We've shown $(aH)(bH) = (bH)(aH)$ holds for any $a, b \in G$, G/H is therefore Abelian.