Student: Yu Cheng (Jade)

Math 412

Homework #5

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Question: Diagonalize these matrices, i.e., find C nonsingular and D diagonal such that $C^{-1}MC = D$.

a. $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Answer: We will look for the eigenvalues and eigenvectors of the given matrix and diagonalize the matrix.

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 - 2\lambda$$

$$\Rightarrow \text{roots of the characteristic polynomial: } \lambda = 0 \text{ or } \lambda = 2\,.$$

We now calculate the eigenvectors, x, of each eigenvalue. When $\lambda = 0$, we have,

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 0 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 - x_2 \\ x_2 - x_1 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{where } a \neq 0\,.$$

When $\lambda = 2$, we have,

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -x_1 - x_2 \\ -x_2 - x_1 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{where } a \neq 0\,.$$

For a $(n \times n)$ matrix, it's diagonalizable if and only if we can find n linearly independent eigenvectors. In other words, the dimension of eigenspace is equal to n. The given matrix A is (2×2) and we found two linearly independent eigenvectors, so A is diagonalizable.

$$\lambda = 0, \qquad x = \left\{ x \in R^2 : x = a\begin{bmatrix}1\\1\end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\lambda = 2, \qquad x = \left\{ x \in R^2 : x = a\begin{bmatrix}1\\-1\end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\Rightarrow C = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$\Rightarrow C^{-1} = \begin{bmatrix}0.5 & 0.5\\0.5 & -0.5\end{bmatrix}$$

$$\Rightarrow D = C^{-1}AC = \begin{bmatrix}0.5 & 0.5\\0.5 & -0.5\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}.$$

b. $B = \begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}$

Answer: We will look for the eigenvalues and eigenvectors of the given matrix and diagonalize the matrix.

$$\lambda I - B = \begin{bmatrix}\lambda - 1 & -3\\-3 & \lambda - 4\end{bmatrix}$$

$$\Rightarrow \det(\lambda I - B) = (\lambda - 1)(\lambda - 4) - 9 = \lambda^2 - 5\lambda - 5$$

$$\Rightarrow \text{roots of the characteristic polynomial: } \lambda = \frac{1}{2}(5 + 3\sqrt{5}) \text{ or } \lambda = \frac{1}{2}(5 - 3\sqrt{5}).$$

We now calculate the eigenvectors, x, of each eigenvalue. When $\lambda = \frac{1}{2}(5 + 3\sqrt{5})$, we have,

$$\begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} - \frac{1}{2}(5 + 3\sqrt{5}) \cdot \begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix}3x_2 - \frac{3}{2}(1 + \sqrt{5})x_1\\3x_1 + \frac{3}{2}(1 - \sqrt{5})x_2\end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix}1\\\frac{1 + \sqrt{5}}{2}\end{bmatrix}, \text{where } a \neq 0.$$

When $\lambda = \frac{1}{2}(5 - 3\sqrt{5})$, we have,

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{1}{2}(5 - 3\sqrt{5}) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3x_2 - \frac{3}{2}(1 - \sqrt{5})x_1 \\ 3x_1 + \frac{3}{2}(1 + \sqrt{5})x_2 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ \frac{1 - \sqrt{5}}{2} \end{bmatrix}, \text{where } a \neq 0\,.$$

For a $(n \times n)$ matrix, it's diagonalizable if and only if we can find n linearly independent eigenvectors. In other words, the dimension of eigenspace is equal to n. The given matrix B is (2×2) and we found two linearly independent eigenvectors, so $B$is diagonalizable.

$$\lambda = \frac{1}{2}(5 + 3\sqrt{5}), \qquad x = \left\{x \in R^2 : x = a\begin{bmatrix} 1 \\ \frac{1 + \sqrt{5}}{2} \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0\right\}$$

$$\lambda = \frac{1}{2}(5 - 3\sqrt{5}), \qquad x = \left\{x \in R^2 : x = a\begin{bmatrix} 1 \\ \frac{1 - \sqrt{5}}{2} \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0\right\}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 1 \\ \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \end{bmatrix}$$

$$\Rightarrow C^{-1} = \begin{bmatrix} \frac{\sqrt{5} - 1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{\sqrt{5} + 1}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\Rightarrow D = C^{-1}BC = \begin{bmatrix} 1 & 1 \\ \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{5} - 1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{\sqrt{5} + 1}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{5 + 3\sqrt{5}}{2} & 0 \\ 0 & \frac{5 - 3\sqrt{5}}{2} \end{bmatrix}.$$

c. $E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Answer: We will look for the eigenvalues and eigenvectors of the given matrix and diagonalize the matrix.

$$\lambda I - E = \begin{bmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 4 & -5 \\ 0 & 0 & \lambda - 6 \end{bmatrix}$$

$$\det(\lambda I - E) = (\lambda - 1)\det\left(\begin{bmatrix} \lambda - 4 & -5 \\ 0 & \lambda - 6 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -5 \\ 0 & \lambda - 6 \end{bmatrix}\right) - 3\det\left(\begin{bmatrix} 0 & \lambda - 4 \\ 0 & 0 \end{bmatrix}\right)$$

$$= (\lambda - 1)(\lambda - 4)(\lambda - 6) + 0 - 0$$

$$= (\lambda - 1)(\lambda - 4)(\lambda - 6)$$

$\Rightarrow$ roots of the characteristic polynomial: $\lambda = 1, \lambda = 4$ or $\lambda = 6$.

We now calculate the eigenvectors, x, of each eigenvalue. When $\lambda = 1$, we have,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2x_2 + 3x_3 \\ 3x_2 + 5x_3 \\ 5x_3 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{where } a \neq 0 .$$

When $\lambda = 4$, we have,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 4 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -3x_1 + 2x_2 + 3x_3 \\ 5x_3 \\ 2x_3 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \text{where } a \neq 0 .$$

When $\lambda = 6$, we have,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 6 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -5x_1 + 2x_2 + 3x_3 \\ -2x_2 + 5x_3 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow x = a \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}, \text{where } a \neq 0 \,.$$

For a $(n \times n)$ matrix, it's diagonalizable if and only if we can find n linearly independent eigenvectors. In other words, the dimension of eigenspace is equal to n. The given matrix E is (3×3) and we found three linearly independent eigenvectors, so E is diagonalizable.

$$\lambda = 1, \qquad x = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\lambda = 4, \qquad x = \left\{ x \in R^3 : x = a \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\lambda = 6, \qquad x = \left\{ x \in R^3 : x = a \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}$$

$$\Rightarrow C^{-1} = \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{15} \\ 0 & \frac{1}{3} & -\frac{5}{16} \\ 0 & 0 & \frac{1}{10} \end{bmatrix}$$

$$\Rightarrow D = C^{-1}EC = \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{15} \\ 0 & \frac{1}{3} & -\frac{5}{16} \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

d. $E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Answer: We will look for the eigenvalues and eigenvectors of the given matrix and diagonalize the matrix.

$$\lambda I - E = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix}$$

$$\det(\lambda I - E) = \lambda \det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\right) + \det\left(\begin{bmatrix} -1 & -1 \\ -1 & \lambda \end{bmatrix}\right) - \det\left(\begin{bmatrix} -1 & \lambda \\ -1 & -1 \end{bmatrix}\right)$$

$$= \lambda(\lambda^2 - 1) + (-\lambda - 1) - (1 + \lambda)$$

$$= \lambda(\lambda - 1)(\lambda + 1) - 2(\lambda + 1)$$

$$= (\lambda + 1)(\lambda^2 - \lambda - 2)$$

$$= (\lambda + 1)^2(\lambda - 2)$$

$\Rightarrow$ roots of the characteristic polynomial: $\lambda = -1$ or $\lambda = 2$.

We now calculate the eigenvectors, x, of each eigenvalue. When $\lambda = -1$, we have,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ or } x = a\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{where } a \neq 0 .$$

When $\lambda = 2$, we have,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 2 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2x_1 + x_2 + x_3 \\ x_1 - 2x_2 + x_3 \\ x_1 + x_2 - 2x_3 \end{bmatrix} = 0$$

$$\Rightarrow x = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{where } a \neq 0 .$$

For a $(n \times n)$ matrix, it's diagonalizable if and only if we can find n linearly independent eigenvectors. In other words, the dimension of eigenspace is equal to n. The given matrix F is (3×3) and we found two linearly independent eigenvectors, so F is diagonalizable.

$$\lambda = -1, \quad x = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\text{or} \quad x = \left\{ x \in R^3 : x = a \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\lambda = 2, \quad x = \left\{ x \in R^3 : x = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{where } a \text{ is any number and } a \neq 0 \right\}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow C^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\Rightarrow D = C^{-1}FC = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$