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Math 612

Homework #1 & #2

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Class Notes, Exercises 2.18

Question 1 Show that $2x^5 - 6x^3 + 9x^2 - 15$ is irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.

Answer: Eisentein Criterion states, if R is a UFD with quotient field F, and there is a polynomial $f(x)$ in $R[x]$, $f(x) = a_n x^n + \cdots + a_1 x + a_0$. If there exists a prime p of R such that p divides all coefficients except the a_n, and p^2 does not divide a_0, then $f(x)$ is irreducible in $F[x]$. If $f(x)$ is primitive then it is also irreducible in $R[x]$.

So, we need to find a prime number $p \in \mathbb{Z}$, such that $p \mid (-15), p \mid 9, p \mid (-6), p \nmid 2$ and $p^2 \nmid (-15)$. Obviously, $p = 3$ satisfies these conditions. Therefore, $f(x) = 2x^5 - 6x^3 + 9x^2 - 15$ is irreducible in $\mathbb{Q}[x]$. At the same time, $gcd(2, -6, 9, -15) = 1$, $C(f(x)) = 1$, a unit, so $f(x)$ is primitive and hence it is irreducible in $\mathbb{Z}[x]$ as well.

Question 2 Let $f = y^3 + x^2y^2 + x^3y + x \in R[x, y]$, where R is a UFD. Show that f is irreducible in $R[x, y]$. (Hint: view $R[x, y]$ as $(R[x])[y]$ and note that x is irreducible, and hence prime in $R[x]$.)

Answer: We can view $R[x, y]$ as $(R[x])[y]$, $f = y^3 + (x^2)y^2 + (x^3)y + x \in R[x, y]$. In a UFD irreducible implies prime and vice versa. $x \in R[x]$ is irreducible, and hence prime.

$$x \mid x,\ x \mid x^3,\ x \mid x^2$$
$$x \nmid 1,\ x^2 \nmid x.$$

So, $f = y^3 + x^2y^2 + x^3y + x$ satisfies the Eisentein Criterion, f is irreducible in $R[x, y]$.

Question 3 Let p be a prime in $\mathbb{Z}$ and $f = \frac{x^p - 1}{x - 1}$. Show that f is irreducible in $\mathbb{Z}[x]$. (Hint: show that $g(x) = f(x + 1)$ is irreducible and use this to show f is irreducible.)

Answer: Let $g(x) = f(x + 1)$,

$$\begin{aligned} g(x) &= \frac{(x+1)^p - 1}{x} \\ &= \frac{1}{x}\left[\sum_{k=0}^{p} \binom{p}{k} x^k - 1\right] \quad \because Binomial's\ Theorem \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x}\left[\binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p-1}x^{p-1} + x^p - 1\right] \\
&= \frac{1}{x}\left[\binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p-1}x^{p-1} + x^p\right] \\
&= \binom{p}{1} + \binom{p}{2}x + \cdots + \binom{p}{p-1}x^{p-2} + x^{p-1} \\
&= \sum_{k=1}^{p}\binom{p}{k}x^{k-1}.
\end{aligned}
$$

Now we can check that $p \in \mathbb{Z}$ is a prime that satisfies the Eisentein Criterion.

$$
\begin{aligned}
\binom{p}{k} = \frac{p!}{k!\cdot(p-k)!} \quad &\Rightarrow \quad p \mid \binom{p}{k}, \forall k \in [1,\ p-1] \\
\binom{p}{p} = 1 \quad &\Rightarrow \quad p \nmid \binom{p}{p} \\
\binom{p}{1} = p \quad &\Rightarrow \quad p^2 \nmid \binom{p}{1}.
\end{aligned}
$$

Therefore, $g(x)$ is irreducible in $\mathbb{Q}[x]$. Since $g(x)$ is a monic, it is primitive. So, $g(x)$ is also irreducible in $\mathbb{Z}[x]$. Assume $f(x)$ is not irreducible in $\mathbb{Z}[x]$, $f(x) = f_1(x)\cdot f_2(x)$, where $f_1(x), f_2(x) \in \mathbb{Z}[x]$. Then we have a ***contradiction***.

$$
\begin{aligned}
f(x) &= f_1(x)\cdot f_2(x) \\
g(x) &= f(x+1) \\
&= f_1(x+1)\cdot f_2(x+1).
\end{aligned}
$$

We found a proper factorization of $g(x)$ in $\mathbb{Z}[x]$, but $g(x)$ is proven to be irreducible in $\mathbb{Z}[x]$. Therefore, $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Class Notes, Exercises 3.2

Question 5 Suppose M_i, $i = 1, 2, \cdots, n$ are submodules of a module M such that the submodule generated by M_i's is all of M that is $M = M_1 + M_2 + \cdots + M_n$. Also assume:

$$
M_i \cap (M_1 + \cdots + M_{i-1} + M_i + \cdots + M_n) \quad = \quad 0.
$$

for$i = 1, \cdots, n$. Show that M is isomorphic to the direct product (or direct sum, they are the same for finite products) of the M_i's.

Answer: We will first show that $M = M_1 + M_2 + \cdots + M_n$ is a module. Let $r, s \in R$, and $x, y \in M$. M is closed under summation:

$$\begin{aligned} x+y &= (x_1+\cdots+x_n)+(y_1+\cdots+y_n) \\ &= (x_1+y_1)+\cdots+(x_n+y_n)\in M. \end{aligned}$$

M is closed under the left multiplication with elements in R:

$$\begin{aligned} rx &= r(x_1+\cdots+x_n) \\ &= rx_1+\cdots+rx_n \\ & \quad rx_i\in M_i \ \because M_i's \ are\ modules \\ &\Rightarrow rx_1+\cdots+rx_n = rx\in M. \end{aligned}$$

Show $r(x+y)=rx+ry$:

$$\begin{aligned} r(x+y) &= r((x_1+\cdots+x_n)+(y_1+\cdots+y_n)) \\ &= r(x_1+\cdots+x_n)+r(y_1+\cdots+y_n) \ \because x_i,\ y_i\, r\in R \\ &= rx+ry. \end{aligned}$$

Show $(r+s)x=rx+sx$:

$$\begin{aligned} (r+s)\,x &= (r+s)(x_1+\cdots+x_n) \\ &= r(x_1+\cdots+x_n)+s(x_1+\cdots+x_n) \ \because x_i,\ y_i\, r\in R \\ &= rx+sx. \end{aligned}$$

Show $(rs)x=r(sx)$:

$$\begin{aligned} (rs)\,x &= (rs)(x_1+\cdots+x_n) \\ &= rsx_1+\cdots+rsx_n \ \because rs,\ x_i\in R \\ r(sx) &= r(s(x_1+\cdots+x_n)) \\ &= r(sx_1+\cdots+sx_n) \ \because x_i,\ s\in R \\ &= rsx_1+\cdots+rsx_n \ \because sx_i,\ r\in R \\ \Rightarrow (rs)\,x &= r(sx)\,. \end{aligned}$$

Now let's define maps: $\phi_i : x_i \to x_i$ from M_i to M, where the left $x_i\in M_i$, the right $x_i\in M$. These maps are clearly homomorphisms. Let's have $a,b\in M_i$, $a,b\in M$ and $r\in R$.

$$\begin{aligned} \phi_i(a+b) &= a+b \\ &= \phi_i(a)+\phi_i(b)\,. \\ \phi_i(ra) &= ra \end{aligned}$$

$$= \quad r\phi_i(a).$$

According to the basic homomorphism property of the direct sum $\oplus_1^n M_i$, there is a ***homomorphism*** ϕ from $\oplus_1^n M_i$ to M. It is defined as:

$$\phi : (x_1, \cdots, x_n) \quad \to \quad \sum_1^n x_i.$$

Now we just need to show this homomorphism is in fact a ***isomorphism***. We will first show that it is ***surjective***. For any $x \in M$, there exist $(x_1, \cdots, x_n) \in \oplus_1^n M_i$ such that $\phi(x_1, \cdots, x_n) = x$. This is true according to the first given condition of M that M is generated by M_i's, $M = M_1 + \cdots + M_n$.

We also need to show that ϕ is ***injective***. For any $x, y \in M$, if $x = y$, then $(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$, where $\phi(x_1, \cdots, x_n) = x$ and $\phi(y_1, \cdots, y_n) = y$. For $(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$, it means $x_i = y_i \forall i \in [1, n]$. Let's assume that not all $x_i = y_i$, and pick out the ones that don't equal, $i \in [k, s], x_i \neq y_i$.

$$\begin{aligned} x &= x_1 + \cdots + x_n \\ y &= y_1 + \cdots + y_n \\ \because x &= y \\ \Rightarrow x_1 + \cdots + x_n &= y_1 + \cdots + y_n \\ \Rightarrow x_k + \cdots + x_s &= y_k + \cdots + y_s \ \ where\ x_i \neq y_i \\ \Rightarrow x_k - y_k &= (x_{k+1} - y_{k+1}) + \cdots + (x_s - y_s) \\ \because x_k \neq y_k &\Rightarrow x_k - y_k = (x_{k+1} - y_{k+1}) + \cdots + (x_s - y_s) \neq 0. \end{aligned}$$

Now we've obtained $x_k - y_k \in M_k$ not being the zero element, and $x_k - y_k$ can be generated by $M_{k+1}, \cdots, M_s$ This is a conflict with the second given condition:

$$M_i \cap (M_1 + \cdots + M_{i-1} + M_i + \cdots + M_n) \quad = \quad 0.$$

Hence $x_i = y_i \forall i \in [1, n]$, and the map is injective. At this point, we've shown there is a isomorphic map from the direct sum $\oplus_1^n M_i$ to M, hence we've shown that M is isomorphic with the direct sum (direct product).

Class Notes, Exercises 3.5

Question 1 Show that if A is an $n-$generated module over a PID and B is a submodule of A, then B can be generated by a set with at most $n$elements.

Answer: Since A is a n generated module, it has a set of n generators. Hence there exist a homomorphism, ϕ, from the free module ${}_RR^n$ to A:

$$\phi : {}_RR^n \to A.$$

B is a submodule of A, $B \leq A$. We can prove that $\phi^{-1}(B)$ forms a submodule of ${}_RR^n$. It is closed under summation, $\phi^{-1}(b_1) + \phi^{-1}(b_2) = \phi^{-1}(b_1 + b_2) \in \phi^{-1}(B)$. It is closed under left multiplication with elements in R, $r \cdot \phi^{-1}(b) = \phi^{-1}(r \cdot b) \in \phi^{-1}(B)$. Hence $\phi^{-1}(B) \leq {}_RR^n$ and ϕ defines an homomorphism from $\phi^{-1}(B)$ to B.

$$\phi : \phi^{-1}(B) \to B.$$

The theorem stated that if R is a PID, any submodule of ${}_RR^n$ is free and has a rank less than or equal to n. Hence, $\phi^{-1}(B)$ is a free module with a rank less than or equal to n. In other words, $\phi^{-1}(B) = {}_RR^m$ where $m \leq n$, since any free module generated by a set of k elements is isomorphic to ${}_RR^k$. Therefore B is generated by at most n elements.

$$\phi : {}_RR^m \to B, \ \ m \leq n.$$

Class Notes, Exercises 3.9

Question 1 Let R be a Euclidean domain and let $SL_n(R)$ be all $n \times n$ matrices over R with determinant 1. Show that the group $SL_n(R)$ is generated by the elementary matrices of the first kind, those of the form $T_{ij}(b) = I_n + bE_{ij}, \ i \neq j$.

Answer: Our goal is to transform a special linear matrix into the identity matrix with only row and columns operations involving elementary matrices of the first kind, $Tij(b)$. In other words, we want to complete this transformation with only shear operations, in which we change a row or column by adding to it a scalar multiplication of another row or column.

First we're going to prove the statement is true if R is a ***field***. Let's have $A \in SL_n(R)$ and $A = (a_{ij})$.

$$A \ = \ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Let's assume that $a_{21} \neq 0$, then we can make $a_{11} = 1$, regardless of the value of a_{11}. This is done by applying a left multiplication of a shear operation to A,

$$\begin{aligned}
T_{12}(b) \cdot A &= \begin{bmatrix} 1 & b & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} 1 & a'_{12} & \cdots & a'_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&\Rightarrow \quad b = \frac{1 - a_{11}}{a_{21}}.
\end{aligned}$$

If $a_{21} = 0$, obviously the equation for b is undefined. Hence we need to be able to transform with shear operations so that $a_{21} \neq 0$ can always be achieved. Since $det(A) = 1$, there exist some $a_{n1} \neq 0$, $\exists a_{x1} \neq 0$.

$$\begin{aligned}
T_{2x}(1) \cdot A &= \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{x1} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \; where\; a_{x1} \neq 0.
\end{aligned}$$

Therefore we can always start with a $a_{21} \neq 0$ and obtain a equivalent matrix with its $a_{11} = 1$.

$$A' = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Now we can show that we are able to eliminate all non-zero values from row 1 and column 1 except $a_{11} = 1$. For instance, for column 1 element, $a_{p1} \neq 0$, we can apply a left multiplication with $T_{p1}(-a_{p1})$ to make it zero.

$$\begin{aligned} T_{p1}\left(-a_{p1}\right) \cdot A' &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_{p1} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \end{aligned}$$

For row 1 element, $a_{1q} \neq 0$, we can apply a right multiplication with $T_{1q}(-a_{1q})$ to make it zero.

$$\begin{aligned} A' \cdot T_{1q}\left(-a_{1q}\right) &= \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & -a_{1q} & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}. \end{aligned}$$

After the above equivalent transformations that get ride of non-zero elements in the first row and first column, we obtain an equivalent matrix of the original special linear matrix in the following form:

$$A'' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Now we can induct on n. The problem is reduced to the bottom right sub-matrix. Hence we just need to show that the base case holds. When $n = 1$, $A = [1]$ since $det(A) = 1$. A is a shear elementary matrix, and hence the base case holds. Therefore, we've shown that through a series of shear transformations we can obtain the identity matrix as an equivalent matrix of $A \in SL_n(R)$. Since all elementary matrices are invertible, and their inverses belong to the same class of elementary matrices.

$$\begin{aligned}
&\Pi T_{ij}(b_k) \cdot A = I_n \\
&\Rightarrow A = \Pi T'_{ij}(b_k) \cdot I_n \\
&\Rightarrow A = \Pi T'_{ij}(b_k).
\end{aligned}$$

At this point, we've shown that the elementary matrices of the first kind generate the $SL_n(R)$ when R is a field, since an arbitrary $A \in SL_n(R)$ can be written as a series of elementary matrices of the first kind.

When R is ***not a field***, the fractions are not defined, we apply the procedure of transforming A to its Smith Normal Form first and then to the identity matrix. We've shown in class that we can transform $A \in GL_n(R)$, where R is an Euclidean Domain (E.D), into its Smith Normal Form with $T_{ij}(b)$ and P_{ij} elementary matrices.

All row or column exchanging operations can be replaced with combinations of scaling operations and shear operations. Hence, the transformation into the Smith Normal Form can be done with $T_{ij}(b)$ and $D_i(-1)$ elementary matrices.

$$D_i(-1) \cdot \begin{bmatrix} \dots & \dots & \dots & \dots \\ a & b & c & \dots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ -a & -b & -c & \dots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$T_{ij}(1) \cdot \begin{bmatrix} \dots & \dots & \dots & \dots \\ -a & -b & -c & \dots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ d-a & e-b & f-c & \dots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$T_{ji}(-1) \cdot \begin{bmatrix} \dots & \dots & \dots & \dots \\ d-a & e-b & f-c & \dots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ d-a & e-b & f-c & \dots \\ \vdots & \vdots & \vdots & \vdots \\ a & b & c & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$
\begin{aligned}
T_{ij}(1)\cdot\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ d-a & e-b & f-c & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a & b & c & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} &= \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ d & e & f & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a & b & c & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \\
&= P_{ij}\cdot\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ a & b & c & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \\
\Rightarrow P_{ij} &= T_{ij}(1)\cdot T_{ji}(-1)\cdot D_i(-1).
\end{aligned}
$$

We notice that we can always pull the scaling to the left side of the sheer when the scaling is $D_i(-1)$. When the row and column involved in the scaling are not involved in the shearing, the multiplication is commutative. For example:

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

When the row and column involved in the scaling is also involved in the shearing, the multiplication is not commutative, but can still be swapped with a modification to the shear matrix. For example:

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & -p \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & -p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Therefore, the series of $T_{ij}(b)$ and $D_i(-1)$ elementary matrices can be written as

$$
\begin{aligned}
&\prod D_i(-1)\cdot\prod T_{ij}(b_k)\cdot A\cdot\prod D_i'(-1)\cdot\prod T_{ij}'(b_k') = A_{Smith-Normal-Form} \\
&\because det\,(D_i(-1)) = -1 \\
&\because det\,(T_{ij}(b_k)) = 1 \\
&\because det\,(A) = 1
\end{aligned}
$$

$$\Rightarrow det\left(A_{Smith-Normal-Form}\right) = 1 \; or \; -1.$$

Since factions are not defined when R is just an E.D, not a field, the invariant factors of $A_{Smith-Normal-Form}$ are either 1 or -1, $d_i \in \{1, -1\}$. In this case, to transform $A_{Smith-Normal-Form}$ to the identity matrix, we simply need to multiply $D_i(-1)$ for the columns (rows) that has $d_i = -1$. Therefore we have

$$\begin{aligned}
&\prod D_i''(-1) \cdot A_{Smith-Normal-Form} \\
&= \prod D_i''(-1) \cdot \prod D_i(-1) \cdot \prod T_{ij}(b_k) \cdot A \cdot \prod D_i'(-1) \cdot \prod T_{ij}'(b_k') \\
&= I_n \\
&\Rightarrow A = \prod_{1}^{m} D_i'''(-1) \cdot \prod T_{ij}''(b_k'') \cdot \\
&\because det\left(A\right) = 1 \\
&\Rightarrow m \in \mathbb{N} \; is \; even.
\end{aligned}$$

Now we just need to show that $D_i(-1) \cdot D_j(-1)$ can be written as a series of multiplications of $T_{ij}(b)$. For example:

$$\begin{aligned}
\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

This conversion is valid in general. In other words, $D_i(-1) \cdot D_j(-1)$ can always be changed into a series of four $T_{ij}(b)$ multiplications.

$$T_{ij}\left(-2\right) \cdot \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ a & b & c & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ a-2d & b-2e & c-2f & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$T_{ji}\left(1\right) \cdot \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ a-2d & b-2e & c-2f & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ d & e & f & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ a-2d & b-2e & c-2f & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a-d & b-e & c-f & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$
\begin{aligned}
T_{ij}\left(-2\right)\cdot\begin{bmatrix}\cdots & \cdots & \cdots & \cdots\\ a-2d & b-2e & c-2f & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ a-d & b-e & c-f & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix} &= \begin{bmatrix}\cdots & \cdots & \cdots & \cdots\\ -a & -b & -c & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ a-d & b-e & c-f & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix}\\
T_{ji}\left(1\right)\cdot\begin{bmatrix}\cdots & \cdots & \cdots & \cdots\\ -a & -b & -c & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ a-d & b-e & c-f & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix} &= \begin{bmatrix}\cdots & \cdots & \cdots & \cdots\\ -a & -b & -c & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ -d & -e & -f & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix}\\
&= D_i(-1)\cdot D_j(-1)\cdot\begin{bmatrix}\cdots & \cdots & \cdots & \cdots\\ a & b & c & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ d & e & f & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix}\\
\Rightarrow D_i(-1)\cdot D_j(-1) &= T_{ji}\left(1\right)\cdot T_{ij}\left(-2\right)\cdot T_{ji}\left(1\right)\cdot T_{ij}\left(-2\right).
\end{aligned}
$$

Now it is clear that we can re-write A into a series of$T_{ij}(b)$ multiplications.

$$
\begin{aligned}
A &= \prod_{1}^{m} D_i'''(-1)\cdot\prod T_{ij}''(b_k'')\ \ \textit{where m is even}\\
\Rightarrow A &= \prod T_{ij}'''(b_k''')\cdot
\end{aligned}
$$

At this point, we've shown that if R is a E.D, but not a field, elementary matrices of the first kind also generate the special linear matrix group. In summary, $T_{ij}(b)$ generate $SL_n(R)$ over an E.D.