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Math 612

Homework #3

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Class Notes, Exercises 2.18

Question 3 Prove that 2×2 matrices over F which are not scalar matrices are similar if and only if they have the same characteristic polynomial.

Answer: We will first prove if two matrices are similar then they have the same characteristic polynomial. Let $A \cong B$. According to the similar matrix definition, we have $A = SBS^{-1}$. The characteristic polynomial of A can be written as the following.

$$\begin{aligned} det\,(xI - A) &= det\left(xI - SBS^{-1}\right) \\ &= det\left(SxIS^{-1} - SBS^{-1}\right) \\ &= det\left[S\,(xI - B)\,S^{-1}\right] \\ &= det\,(S) \cdot det\,(xI - B) \cdot det\left(S^{-1}\right) \\ &= det\,(S) \cdot det\left(S^{-1}\right) \cdot det\,(xI - B) \\ &= det\left(SS^{-1}\right) det\,(xI - B) \\ &= det\,(xI - B)\,. \end{aligned}$$

Hence we've shown that the similar matrices share the same characteristic polynomial. Now we need to to prove that if A, B have the same characteristic polynomial and they are 2×2 matrices, then they are similar to each other. Let's have the characteristic polynomial be $char_A(x) = x^2 + ax + b$ where $a, b \in F$,

First let's assume that $char_A(x) = x^2 + ax + b$ is ***a prime polynomial*** in $F[x]$. In this case, the minimal polynomials of A, B are the same, $m_A(x) = m_B(x) = x^2 + ax + b$. Then we can write the Smith Normal Form (SNF) for the x matrices of A, B as the following.

$$\begin{aligned} SNF_A &= SNF_B \\ &= \begin{bmatrix} 1 & 0 \\ 0 & x^2 + ax + b \end{bmatrix}. \end{aligned}$$

Apparently, they have the same Smith Normal Form. A, B are therefore similar. Now let's consider the case when $char_A(x) = x^2 + ax + b$ is ***not a prime polynomial.*** If $char_A(x) =$

$x^2 + ax + b = (x + e)(x + f)$, where $e \neq f$, then we still have the following.

$$\begin{aligned} SNF_A &= SNF_B \\ &= \begin{bmatrix} 1 & 0 \\ 0 & (x+e)(x+f) \end{bmatrix}. \end{aligned}$$

Now, let's assume $char_A(x) = x^2 + ax + b = (x+e)^2$ for some $e \in F$. We also assume A, B have different minimal polynomials. WLOG let's have $m_A(x) = x + e$ and $m_B(x) = (x + e)^2$. So, $A \vDash x + e$, it follows that A is a scalar matrix. Scalar matrices are excluded from the problem. In other words, $A \nvDash x + e$ and $m_A(x) \neq x + e$, hence, $m_A(x) = m_B(x) = (x + e)^2 = char_A(x)$.

So whether or not $char_A(x) = x^2 + ax + b$ is a factorisable, we have $m_A(x) = m_B(x) = char(x)$, and hence SNFs of A, B are the same. At this point, we've proven both sides, and we conclude that non-scalar 2×2 matrices are similar to each other if and only if they have the same characteristic polynomial.

Question 4 Prove that two 3×3 matrices are similar if and only if they have the same characteristic and same minimal polynomials. Give an explicit counterexample to this assertion for 4×4 matrices.

Answer: We've shown in the previous question that if two matrices are similar, then they have the same characteristic polynomials. It is clear two similar matrices have the same minimal polynomial as well, since they have the same SNF for their x matrices. So we've shown one direction.

Let two 3×3 matrices A, B have the same characteristic polynomial, $char_A(x)$, and the same minimal polynomial, $m_A(x)$. If $m_A(x)$ has a degree of 1, then we have only one possible SNF.

$$SNF(xI - A) = \begin{bmatrix} m_A(x) & 0 & 0 \\ 0 & m_A(x) & 0 \\ 0 & 0 & m_A(x) \end{bmatrix}.$$

If $m_A(x)$ has a degree of 2, then we have only one possible SNF.

$$SNF(xI - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & char_A(x)/m_A(x) & 0 \\ 0 & 0 & m_A(x) \end{bmatrix}.$$

If $m_A(x)$ has a degree of 3, then we still have only one possible SNF.

$$SNF(xI - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m_A(x) \end{bmatrix}.$$

In summary, for a 3×3 matrix, the characteristic polynomial and the minimal polynomial determine its x matrix's SNF. Two 3×3 matrices are similar if and only if they share both the characteristic polynomial and the minimal polynomial.

A counter example can be derived as the following. Let's have $char_A(x) = (x-1)^4$ and $m_A(x) = (x-1)^2$. Then we can have the two different kind of SNFs.

$$
\begin{aligned}
SNF_1(xI - A) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (x-1)^2 & 0 \\ 0 & 0 & 0 & (x-1)^2 \end{bmatrix} \\
SNF_2(xI - A) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & (x-1)^2 \end{bmatrix}.
\end{aligned}
$$

First we compute the Rational Canonical Matrices (RCMs) for polynomials presented in the SNFs.

$$
\begin{aligned}
RCM\,(x-1) &= [1] \\
RCM\left((x-1)^2\right) &= RCM\left(x^2 - 2x + 1\right) \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.
\end{aligned}
$$

With these two SNFs and the RCMs, we can derive two different Rational Canonical Forms (RCFs).

$$
\begin{aligned}
SNF_1(xI - A) \;&\Rightarrow\; RCF_1(A) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
SNF_2(xI - A) \;&\Rightarrow\; RCF_2(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.
\end{aligned}
$$

These two matrices share the same characteristic polynomial, $char_A(x) = (x-1)^4$, and the same minimal polynomial, $m_A(x) = (x-1)^2$, but they are not similar to each other.

Question 6 Prove that the constant term in the characteristic polynomial of the $n \times n$ matrix A is $(-1)^n det(A)$ and that the coefficient of x^{n-1} is the negative of the sum of the diagonal entries of A (the sum of the diagonal entries of A is called the trace of A). Prove that the $det(A)$ is the product of the eigenvalues of A and that the trace of A is the sum of the eigenvalues of A.

Answer: *First*, we'll show the constant term, c, in the characteristic polynomial of A is $(-1)^n det(A)$. Let's call the characteristic polynomial $char_A(x)$. We have the following equality.

$$\begin{aligned} det(xI - A) &= char_A(x) \\ char_A(0) &= c \\ \Rightarrow det(-A) &= c. \end{aligned}$$

Now, the problem comes down to proving $det(-A)$ is $(-1)^n det(A)$. This is easy, since we know that the scale elementary matrix $D_i(b)$ affects the determinant by introducing a scalar term b.

$$det\left(D_i(b)A\right) = b \cdot det(A).$$

From A to $-A$, we've introduced n times of scale elementary operations, $D_0(-1),\ D_1(-1), \cdots, D_n(-1)$.

$$\begin{aligned} -A &= D_0(-1)D_1(-1)\cdots D_n(-1)A \\ \Rightarrow det(-A) &= (-1)^n det(A). \end{aligned}$$

Second, we'll show that the coefficient of x^{n-1} is the negative of the sum of the diagonal entries of A. The sum of the diagonal entries is called the trace of A. Let's exam the $xI - A$ matrix.

$$\begin{aligned} A &= \begin{bmatrix} a & \cdots & b \\ \vdots & \ddots & \vdots \\ c & \cdots & d \end{bmatrix} \\ xI - A &= \begin{bmatrix} x-a & \cdots & -b \\ \vdots & \ddots & \vdots \\ -c & \cdots & x-d \end{bmatrix} \end{aligned}$$

We'll prove by induction. When $n = 2$, the to be proven statement holds.

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ det\left(xI - A\right) &= det\left(\begin{bmatrix} x-a & -b \\ -c & x-d \end{bmatrix}\right) \\ &= x^2 - (a+d)\,x - bc. \end{aligned}$$

Let's assume for a $k \times k$ matrix A the coefficient of x^{k-1} term in $char_A(x)$ is the negated trace of A.

$$
\begin{aligned}
A &= \begin{bmatrix} a_1 & \cdots & a_k \\ \vdots & \ddots & \vdots \\ b_1 & \cdots & b_k \end{bmatrix} \\
det\,(xI - A) &= det\left(\begin{bmatrix} x - a_1 & \cdots & -a_k \\ \vdots & \ddots & \vdots \\ -b_1 & \cdots & x - b_k \end{bmatrix}\right) \\
&= x^k - (a_1 + \cdots + b_k)\, x^{k-1} + \cdots .
\end{aligned}
$$

Now we want to show extending A by one row and one column the given statement still holds.

$$
\begin{aligned}
B \quad &= \quad \begin{bmatrix} c_1 & \cdots & \cdots & \cdots \\ \vdots & a_1 & \cdots & a_k \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & b_1 & \cdots & b_k \end{bmatrix} \\
det\,(xI - B) \quad &= (x - c_1) \cdot \quad det\left(\begin{bmatrix} x - a_1 & \cdots & -a_k \\ \vdots & \ddots & \vdots \\ -b_1 & \cdots & x - b_k \end{bmatrix}\right) + f(x) \\
&= \quad (x - c_1) \cdot \left(x^k - (a_1 + \cdots + b_k)\, x^{k-1} + \cdots\right) + f(x) \\
&= \quad \left[x^{k+1} - (c_1 + a_1 + \cdots + b_k)\, x^k + \cdots\right] + f(x).
\end{aligned}
$$

Notice that $f(x)$ in the equations above is obtained from the row/column expansion when looking for the determinant of $xI - B$. Since exactly one row/column containing x is crossed out for each term in $f(x)$, the highest order in $f(x)$ is $k - 1$. Therefore $f(x)$ does not affect the coefficient of the x^k term. Hence we've proven by induction that the coefficient of the second highest term in the characteristic polynomial is the negated $tr(A)$.

<u>Third</u>, we'll prove that $det(A)$ is the product of the eigenvalues of A. This statement directly follow the proof that the constant term, c, in the characteristic polynomial is $-det(A)$. Since eigenvalues of A are the roots of $char_A(x)$, we can write $char_A(x)$ as the following. It might contain repeats of λ_i.

$$
\begin{aligned}
char_A(x) &= (x - \lambda_1) \cdot (x - \lambda_2) \cdot \cdots \cdot (x - \lambda_n)\,. \\
\Rightarrow char_A(x) &= x^n + \cdots + ((-\lambda_1) \cdot (-\lambda_2) \cdot \cdots \cdot (-\lambda_n)) \\
\Rightarrow c &= (-1)^n\, \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n \\
\because c &= (-1)^n\, det(A)
\end{aligned}
$$

$$\Rightarrow det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$$

Finally, we'll prove that the trace of A is the sum of the eigenvalues of A. This statement directly follow the proof that the coefficient of x^{n-1} is the negated trace.

$$\begin{aligned}
char_A(x) &= (x-\lambda_1)\cdot(x-\lambda_2)\cdot\dots\cdot(x-\lambda_n)\,. \\
\Rightarrow char_A(x) &= x^n - (\lambda_1+\lambda_2+\dots+\lambda_n)\,x^{n-1} + \dots + ((-\lambda_1)\cdot(-\lambda_2)\cdot\dots\cdot(-\lambda_n)) \\
\Rightarrow -tr(A) &= -(\lambda_1+\lambda_2+\dots+\lambda_n) \\
\Rightarrow tr(A) &= \lambda_1+\lambda_2+\dots+\lambda_n.
\end{aligned}$$

Question 7 Determine the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Answer: We can easily determine that the determinant of this matrix is 1 by executing the row expansion algorithm on the last row. The trace of the matrix is 0. According to the conclusions proved in the previous question, we have the following constraints for the characteristic polynomial and the eigenvalues.

$$\begin{aligned}
char_A(x) &= x^4 + ax^2 - 1 \\
\lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\lambda_4 &= 1 \\
\lambda_1+\lambda_2+\lambda_3+\lambda_4 &= 0
\end{aligned}$$

We can verify this by computing these values.

$$\begin{aligned}
det(xI-A) &= det\left(\begin{bmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ -1 & 0 & 0 & x \end{bmatrix}\right) \\
&= x\cdot det\left(\begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \\ 0 & 0 & x \end{bmatrix}\right) + det\left(\begin{bmatrix} 0 & -1 & 0 \\ 0 & x & -1 \\ -1 & 0 & x \end{bmatrix}\right) \\
&= x^4 - 1 \\
&\Rightarrow \lambda_1,\ \lambda_2 = \pm 1 \\
&\quad\ \lambda_3,\ \lambda_4 = \pm i.
\end{aligned}$$

Question 10 Find all similarity classes of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $(x+2)^2(x-1)$. It suffices to give all lists of invariant factors and write out some of their corresponding matrices.

Answer: We know that the minimal polynomial of a matrix covers all prime divisors of the characteristic polynomial and divides the characteristic polynomial. Also, the degree of the characteristic polynomial is the size of the matrix. So, we derive the following possible characteristic polynomials:

$$\begin{array}{ll} 1. & (x+2)^5(x-1) \\ 2. & (x+2)^4(x-1)^2 \\ 3. & (x+2)^3(x-1)^3 \\ 4. & (x+2)^2(x-1)^4. \end{array}$$

With these possible characteristic polynomials and the fact that the minimal polynomial is the largest invariant factor, we can derive the possible sets of invariant factors.

$$\begin{array}{rcl} char_A(x) = (x+2)^5(x-1) & \Rightarrow & 1.\ \ \{d_i(x)\} = \big\{x+2,\ (x+2)^2,\ (x+2)^2(x-1)\big\} \\ & & 2.\ \ \{d_i(x)\} = \big\{x+2,\ x+2,\ x+2,\ (x+2)^2(x-1)\big\} \\ char_A(x) = (x+2)^4(x-1)^2 & \Rightarrow & 3.\ \ \{d_i(x)\} = \big\{(x+2)^2(x-1),\ (x+2)^2(x-1)\big\} \\ & & 4.\ \ \{d_i(x)\} = \big\{x+2,\ (x+2)(x-1),\ (x+2)^2(x-1)\big\} \\ char_A(x) = (x+2)^3(x-1)^3 & \Rightarrow & 5.\ \ \{d_i(x)\} = \big\{x-1,\ (x+2)(x-1),\ (x+2)^2(x-1)\big\} \\ char_A(x) = (x+2)^2(x-1)^4 & \Rightarrow & 6.\ \ \{d_i(x)\} = \big\{x-1,\ x-1,\ x-1,\ (x+2)^2(x-1)\big\}. \end{array}$$

With these invariant factors we can construct the RCFs, each of which represents a similar class of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $(x+2)^2(x-1)$. Two examples of such similarity classes are constructed and shown below.

$$\{d_i(x)\} = \big\{x+2,\ (x+2)^2,\ (x+2)^2(x-1)\big\} = \big\{x+2,\ x^2+4x+4,\ x^3+3x^2-4\big\}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\{d_i(x)\} = \big\{x+2,\ x+2,\ x+2,\ (x+2)^2(x-1)\big\} = \big\{x+2,\ x+2,\ x+2,\ x^3+3x^2-4\big\}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

Question 14 Determine all possible RCFs for a linear transformation with characteristic polynomial $x^2\left(x^2+1\right)^2$.

Answer: With the given characteristic polynomial, $x^2\left(x^2+1\right)^2$, there are several possible choices of the minimal polynomial.

$$\begin{array}{ll} 1. & m_A(x) = x^2\left(x^2+1\right)^2 \\ 2. & m_A(x) = x\left(x^2+1\right)^2 \\ 3. & m_A(x) = x^2\left(x^2+1\right) \\ 4. & m_A(x) = x\left(x^2+1\right). \end{array}$$

We have the following possible lists of invariant factors.

$$\begin{aligned} m_A(x) = x^2\left(x^2+1\right)^2 &\Rightarrow 1.\ \{d_i(x)\} = \left\{x^2\left(x^2+1\right)^2\right\} \\ m_A(x) = x\left(x^2+1\right)^2 &\Rightarrow 2.\ \{d_i(x)\} = \left\{x,\ x\left(x^2+1\right)^2\right\} \\ m_A(x) = x^2\left(x^2+1\right) &\Rightarrow 3.\ \{d_i(x)\} = \left\{x^2+1,\ x^2\left(x^2+1\right)\right\} \\ m_A(x) = x\left(x^2+1\right) &\Rightarrow 4.\ \{d_i(x)\} = \left\{x\left(x^2+1\right),\ x\left(x^2+1\right)\right\}. \end{aligned}$$

With these invariant factors we can construct the RCFs of the linear transformation.

$$\begin{aligned} \{d_i(x)\} = \left\{x^2\left(x^2+1\right)^2\right\} &= \left\{x^6+2x^4+x^2\right\} \\ &\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \{d_i(x)\} = \left\{x,\ x\left(x^2+1\right)^2\right\} &= \left\{x,\ x^5+2x^3+x\right\} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\{d_i(x)\} = \left\{x^2+1,\ x^2\left(x^2+1\right)\right\} = \left\{x^2+1,\ x^4+x^2\right\}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\{d_i(x)\} = \left\{x\left(x^2+1\right),\ x\left(x^2+1\right)\right\} = \left\{x^3+x,\ x^3+x\right\}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Question 15 Determine up to similarity all 2×2 rational matrices (i.e., $\in M_2(\mathbb{Q})$) of precise order 4 (multiplicatively). Do the same if the matrix has entries from $\mathbb{C}$.

Answer: Matrix A has order4, so $A \vDash x^4 - 1$. Since the minimal polynomial of A divides every polynomial of which A satisfies. We have the following possibilities of $m_A(x)$ over $\mathbb{Q}$.

$$\begin{aligned} A &\vDash x^4-1 = \left(x^2+1\right)(x+1)(x-1) \\ &\Rightarrow m_A(x) \in \left\{x^2+1,\ x+1,\ x-1\right\} \\ &or\ \ m_A(x) \in \left\{\left(x^2+1\right)(x+1),\ \left(x^2+1\right)(x-1),\ x^2-1\right\} \\ &or\ \ m_A(x) \in \left\{x^4-1\right\}. \end{aligned}$$

We are also given that$A$ has a precise order of 4, hence, $A \nvDash x^3-1$, $A \nvDash x^2-1$ and $A \nvDash x-1$. In other words, we have the following relationships.

$$\begin{aligned} x^3-1 = (x-1)\left(x^2+x+1\right) &\neq 0 \\ x^2-1 = (x+1)(x-1) &\neq 0 \\ x-1 &\neq 0. \end{aligned}$$

Since $A \vDash m_A(x)$, by excluding $x-1, x+1$, and x^2-1 we can narrow down the choices of $m_A(x)$ to the following.

$$m_A(x) \in \left\{x^2+1,\ \left(x^2+1\right)(x+1),\ \left(x^2+1\right)(x-1),\ x^4-1\right\}.$$

The degree of $m_A(x)$ is smaller than the degree of the characteristic polynomial of A, which is 2, hence we can narrow down the choice of $m_A(x)$ even more. Actually, only one left.

$$\begin{aligned} & m_A(x) = x^2+1 \\ \Rightarrow\ & SNF\,(xI-A) = \begin{bmatrix} 1 & 0 \\ 0 & x^2+1 \end{bmatrix} \\ \Rightarrow\ & RNF(A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Let's do the same with 2×2 matrices over $\mathbb{C}$. Since we can factor x^4-1 further, we have more candidates for $m_A(x)$.

$$\begin{aligned} A \quad \vDash \quad & x^4-1 = (x+i)(x-i)(x+1)(x-1) \\ \Rightarrow \quad & m_A(x) \in \{x+i,\ x-i,\ x+1,\ x-1\} \\ \textit{or} \quad & m_A(x) \in \left\{x^2+1,\ (x+i)(x+1),\ (x+i)(x-1),\ (x-i)(x+1),\ (x-i)(x-1),\ x^2-1\right\} \\ \textit{or} \quad & m_A(x) \in \left\{\left(x^2+1\right)(x+1),\ \left(x^2+1\right)(x-1),\ (x+i)\left(x^2-1\right),\ (x-i)\left(x^2-1\right)\right\}. \end{aligned}$$

Since $A \vDash m_A(x)$, by excluding $x-1, x+1$, and x^2-1 we can narrow down the choices of $m_A(x)$. Also the degree of $m_A(x)$ cannot be more than 2 as argued previously.

$$\begin{aligned} & m_A(x) \in \{x+i,\ x-i\} \\ \textit{or} \quad & m_A(x) \in \left\{x^2+1,\ (x+i)(x+1),\ (x+i)(x-1),\ (x-i)(x+1),\ (x-i)(x-1)\right\}. \end{aligned}$$

With these 7 minimal polynomials, we can derive 7 sets of invariant factors. With the derived invariant factor sets, we can derive all similarity classes.

$$\begin{aligned} m_A(x) \quad &= \quad x+i \\ & \Rightarrow\ SNF(xI-A) = \begin{bmatrix} x+i & 0 \\ 0 & x+i \end{bmatrix} \\ & \Rightarrow\ RNF(A) = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} \\ m_A(x) \quad &= \quad x-i \\ & \Rightarrow\ SNF(xI-A) = \begin{bmatrix} x-i & 0 \\ 0 & x-i \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& \Rightarrow \quad RNF(A) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \\
m_A(x) \quad & = \quad x^2+1 \\
& \Rightarrow \quad SNF(xI-A) = \begin{bmatrix} 1 & 0 \\ 0 & x^2+1 \end{bmatrix} \\
& \Rightarrow \quad RNF(A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
m_A(x) \quad & = \quad (x+i)(x+1) \\
& \Rightarrow \quad SNF(xI-A) = \begin{bmatrix} 1 & 0 \\ 0 & (x+i)(x+1) \end{bmatrix} \\
& \Rightarrow \quad RNF(A) = \begin{bmatrix} 0 & -i \\ 1 & -(i+1) \end{bmatrix} \\
m_A(x) \quad & = \quad (x+i)(x-1) \\
& \Rightarrow \quad SNF(xI-A) = \begin{bmatrix} 1 & 0 \\ 0 & (x+i)(x-1) \end{bmatrix} \\
& \Rightarrow \quad RNF(A) = \begin{bmatrix} 0 & i \\ 1 & 1-i \end{bmatrix} \\
m_A(x) \quad & = \quad (x-i)(x+1) \\
& \Rightarrow \quad SNF(xI-A) = \begin{bmatrix} 1 & 0 \\ 0 & (x-i)(x+1) \end{bmatrix} \\
& \Rightarrow \quad RNF(A) = \begin{bmatrix} 0 & i \\ 1 & i-1 \end{bmatrix} \\
m_A(x) \quad & = \quad (x-i)(x-1) \\
& \Rightarrow \quad SNF(xI-A) = \begin{bmatrix} 1 & 0 \\ 0 & (x-i)(x-1) \end{bmatrix} \\
& \Rightarrow \quad RNF(A) = \begin{bmatrix} 0 & -i \\ 1 & i+1 \end{bmatrix}.
\end{aligned}$$

In summary, there is 1 similarity class of 2×2 matrices with precise order 4 over $\mathbb{Q}$. That is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. There are 7 similarity classes of 2×2 matrices with precise order 4 over $\mathbb{C}$. They are $\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$, $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -i \\ 1 & -(i+1) \end{bmatrix}$, $\begin{bmatrix} 0 & i \\ 1 & 1-i \end{bmatrix}$, $\begin{bmatrix} 0 & i \\ 1 & i-1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -i \\ 1 & i+1 \end{bmatrix}$.