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Math 612

Midterm

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Class Notes, Exercises 3.20

Question 1 If ρ is a retraction of A onto B, then $\rho\mid_B$, ρ restricted to B, is the identity on B.

Answer: A homomorphism $\rho : A \to A$ is a retraction if $\rho \circ \rho = \rho$. We call B a retract of A if $img(\rho) = B$. It is clear that B is a subalgebra of A, since $B = img(\rho) \subseteq A$. Now we construct the map $\rho\mid_B$ by taking ρ and restrict it to B,

$$\rho\mid_B \quad : \quad B \to B.$$

We will first prove that this map is a ***surjective***. In other words, for any $b \in B$, we want to show there exist $b' \in B$, such that $\rho(b') = b$.

$$\begin{aligned}
&\because \quad B = img\left(\rho\mid_A\right)\\
&\therefore \quad \forall b \in B,\ \exists a \in A,\ \rho\left(a\right) = b\\
&\because \quad \rho\left(a\right) = \rho\left(\rho\left(a\right)\right) = \rho\left(b'\right)\\
&\therefore \quad \forall b \in B,\ \exists b' \in B,\ \rho\left(b'\right) = b.
\end{aligned}$$

Now we will show that it is also ***injective***. For any $b_1, b_2 \in B,\ \rho\left(b_1\right) = \rho\left(b_2\right)$, we want to show that $b_1 = b_2$.

$$\begin{aligned}
&\because \quad \rho\mid_B \text{ is surjective}\\
&\therefore \quad \exists b_1',\ b_2',\ \rho\left(b_1'\right) = b_1,\ \rho\left(b_2'\right) = b_2\\
&\Rightarrow \quad b_1 = \rho\left(b_1'\right) = \rho\left(\rho\left(b_1'\right)\right) = \rho\left(b_1\right)\\
&\qquad\ \ b_2 = \rho\left(b_2'\right) = \rho\left(\rho\left(b_2'\right)\right) = \rho\left(b_2\right)\\
&\because \quad \rho\left(b_1\right) = \rho\left(b_2\right)\\
&\therefore \quad b_1 = b_2.
\end{aligned}$$

At this point, we've shown that $\rho\mid_B\colon B \to B$ is an isomorphism, and hence ***automorphism*** from B to itself. During the proof of injectivity, we've shown that this is an identity map.

$$b = \rho\left(b'\right) = \rho\left(\rho\left(b'\right)\right) = \rho\left(b\right).$$

Question 2 Prove the following are equivalent for an algebra P in a variety $\mathcal{V}$:

1. P is projective.
2. If $f : A \twoheadrightarrow P$ is an epimorphism then there is a homomorphism $g : P \to A$ so that $fg(x) = x$. Note this forces g to be a monomorphism.
3. P is isomorphic to a retract of a free algebra in $\mathcal{V}$.

Answer: We will first show *property 1 implies property 2*. An algebra P is projective if for algebra $M, N \in \mathcal{V}$, there exist an epimorphism $f : M \twoheadrightarrow N$ and a homomorphism $h : P \to N$, then there exist a homomorphism $g : P \to M$ with $h = f \circ g$.

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "h"] \\ M \arrow[r, two heads, "f"'] & N \end{tikzcd}$$

Let $M = A$ and $N = P$, in property 2, we have $f : A \twoheadrightarrow P$. And we can easily construct a homomorphism from $P \to P$, the identity map, id_P. So, we've obtained the following relation:

$$\begin{tikzcd} & P \arrow[d, "id_P"] \\ A \arrow[r, two heads, "f"'] & P \end{tikzcd}$$

According to property 1 that P is projective, we can conclude that there exist a homomorphism $g : P \to A$. So the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "id_P"] \\ A \arrow[r, two heads, "f"'] & P \end{tikzcd}$$

Now, by these homomorphisms, we have $f \circ g = id_P$. In other words, $f \circ g(x) = x$. We've proved property 2 giving property 1.

Now we will show *property 2 implies property 3*. Every module P is the quotient of a free module, e.g., the free module, $\mathcal{F}$, on the set of elements in P. So there is always an exact sequence $0 \to ker\varphi \to \mathcal{F} \overset{\varphi}{\to} P \to 0$. According to property 2, we know there is a homomorphism $g : P \to \mathcal{F}$ and $\varphi \circ g = id_P$.

According to *proposition 25* of Dummit & Foot, a short exact sequence, $0 \to A \to B \overset{\phi}{\to} C \to 0$, is split if and only if there exist a homomorphism $\mu : C \to B$ such that $\phi \circ \mu = id_C$. Hence, $0 \to ker\varphi \to \mathcal{F} \to P \to 0$ is split. Therefore, $\mathcal{F} \cong ker\varphi \oplus P$.

We also have $P \cong 1 \oplus P$. We can construct a homomorphism, $\pi : ker\varphi \oplus P \to 1 \oplus P$ with $\pi(k, p) = (1, p), \forall k \in ker\varphi$. Showing with a diagram it looks like the following.

$$\begin{array}{ccccccccc} & & P & & & & & 1 \oplus P \\ & {}^{g}\swarrow & \downarrow^{id_P} & \Leftrightarrow & & & {}^{g}\swarrow & \downarrow^{id_P} \\ \mathcal{F} & \underset{\pi}{\twoheadrightarrow} & P & & & ker\varphi \oplus P & \underset{\pi}{\twoheadrightarrow} & 1 \oplus P \end{array}$$

With this homomorphism, we can see that $ker\varphi \oplus P$ is a retraction, and $1 \oplus P$ is its retract.

$$\begin{aligned} \because \quad & \pi\left(\pi\left(k, p\right)\right) = \pi\left(1, p\right) \\ & \pi\left(k, p\right) = \left(1, p\right) = \pi\left(1, p\right) \\ \Rightarrow \quad & \pi\left(\pi\left(k, p\right)\right) = \pi\left(k, p\right) \\ \Rightarrow \quad & \pi \circ \pi = \pi. \end{aligned}$$

So $P \cong 1 \oplus P$ is a retract of $\mathcal{F} \cong ker\varphi \oplus P$ with $\pi \circ \pi = \pi$. We've proved property 3 giving property 2.

Now we need to show *property 3 implies property 1*. In other words, given P is isomorphic to a retract of a free algebra $\mathcal{F}$ in $\mathcal{V}$, we want to show that if there is an epimorphism $\alpha : M \twoheadrightarrow N$ and a homomorphism $\beta : P \to N$ then there exist a homomorphism $\gamma : P \to M$.

We start with the following relationship, where $\pi : \mathcal{F} \to P$, P is the retract of $\mathcal{F}$, $\alpha : M \twoheadrightarrow N$ and $\beta : P \to N$

$$\begin{array}{ccc} ker\varphi \oplus P \cong \mathcal{F} & \overset{\pi}{\twoheadrightarrow} & P \\ & & \downarrow^{\beta} \\ M & \overset{\alpha}{\twoheadrightarrow} & N \end{array}$$

According to the universal mapping property, there is a unique homomorphism $\delta : \mathcal{F} \to M$. So we have the following diagram commutes, $\alpha \circ \delta = \beta \circ \pi$

$$\begin{array}{ccc} ker\varphi \oplus P \cong \mathcal{F} & \overset{\pi}{\twoheadrightarrow} & P \\ {}^{\delta}\downarrow & & \downarrow^{\beta} \\ M & \overset{\alpha}{\twoheadrightarrow} & N \end{array}$$

Let's define a map $\gamma : P \to M$ by $\gamma\left(x\right) = \delta\left(\left(0, x\right)\right)$, where $\left(0, x\right) \in ker\varphi \oplus P$.

$$\begin{array}{ccc} ker\varphi \oplus P \cong \mathcal{F} & \overset{\pi}{\twoheadrightarrow} & P \\ {}^{\delta}\downarrow & {}^{\gamma}\swarrow & \downarrow^{\beta} \\ M & \overset{\alpha}{\twoheadrightarrow} & N \end{array}$$

$$\alpha \circ \gamma(x) = \alpha \circ \delta((0,x)) = \beta \circ \pi((0,x)) = \beta(x)$$
$$\Rightarrow \alpha \circ \gamma = \beta.$$

Therefore we've shown there exist a homomorphism $\gamma : P \to M$ and hence P is projective.

$$\begin{array}{ccc} & & P \\ & {}^{\gamma}\swarrow & \downarrow^{\beta} \\ M & \overset{\alpha}{\twoheadrightarrow} & N \end{array}$$

At this point, we've shown these three properties are equivalent.

$$\begin{array}{ccccc} & & \text{propertiy 1} & & \\ & \nearrow & & \searrow & \\ \text{propertiy 3} & & \leftarrow & & \text{propertiy 2} \end{array}$$

Question 3 Let R be a ring and assume now that $\mathcal{V}$ is the variety of all R-modules.

a. Show that if A is and B are R-modules and if $\rho : A \to B$ is a retraction, then A has a submodule C such that $A = B \oplus C$ and $\rho(b, c) = b$.

Answer: Let's have $C = ker\rho$ and construct the following sequence by defining $\varrho : ker\rho \to A$ with $\varrho(x) = x$, and $\rho : A \to B$. We will show that this sequence is a short exact sequence.

$$0 \to ker\rho \overset{\varrho}{\to} A \overset{\rho}{\to} B \to 0.$$

$\varrho : ker\rho \to A$ is injective by definition. $\rho : A \to B$ is surjective since $B = img\rho$. For any $b \in B$ there exist $\rho(a) = b$. We also have $img\varrho = ker\rho$ since ϱ is constructed in such a way. Hence we've shown that the above sequence is a short exact sequence.

As we've proved in ***Question 1***, If ρ is a retraction of A onto B, then ρ restricted to B is the identity map on B. Hence, $\exists \rho' : B \to A$, with $\rho'(b) = b$. Now, we have the following relation.

$$\begin{aligned} & \rho \circ \rho'(b) = \rho(b) = b \\ \Rightarrow \quad & \rho \circ \rho'(b) = id_B. \end{aligned}$$

According to ***proposition 25*** of Dummit & Foot, a short exact sequence, $0 \to X \to Y \overset{\phi}{\to} Z \to 0$, is split if and only if there exist a homomorphism $\mu : Z \to Y$ such that $\phi \circ \mu = id_Z$. So we've shown that $0 \to ker\rho \overset{\varrho}{\to} A \overset{\rho}{\to} B \to 0$ is split, $A = B \oplus ker\rho$, ρ corresponds to an identity map on B, and hence is the nature projection with $\rho(b, c) = b$.

b. Use this to show that an R-module is projective iff it is a direct summand of a free R-module.

Answer: According to *Question 2*, P is isomorphic to a retract of a free algebra in $\mathcal{V}$ if and only if P is projective. If P is isomorphic to a retract of a free R-module $\mathcal{F}$, then according to the conclusion of *part a* of this question, P is a direct summand of $\mathcal{F}$. So, we've shown the only if direction.

If P is a direct summand of a free R-module $\mathcal{F}$, then we have epimorphism $\phi : \mathcal{F} \twoheadrightarrow P$ defined as the nature projection map, $\phi((p, 1, \cdots, 1)) = p$. We also have a monomorphism $\varphi : P \rightarrowtail \mathcal{F}$ defined as $\varphi(p) = (p, 1, \cdots, 1)$.

$$\begin{tikzcd} & P \arrow[dl, "\varphi"'] \arrow[d, "id"] \\ \mathcal{F} \arrow[r, two heads, "\phi"'] & P \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} & P \oplus 1 \oplus \cdots \oplus 1 \arrow[dl, "\varphi'"'] \arrow[d, "id"] \\ P \oplus Q_1 \oplus \cdots \oplus Q_n \arrow[r, two heads, "\phi'"'] & P \oplus 1 \oplus \cdots \oplus 1 \end{tikzcd}$$

On the right-side diagram, we define $\phi' : P \oplus Q_1 \oplus \cdots \oplus Q_n \to P \oplus 1 \oplus \cdots \oplus 1$ as $\phi'((p, 1, \cdots, 1)) = (p, 1, \cdots, 1)$, and $\varphi' : P \oplus 1 \oplus \cdots \oplus 1 \to P \oplus Q_1 \oplus \cdots \oplus Q_n$ as $\varphi'((p, 1, \cdots, 1)) = (p, 1, \cdots, 1)$, clearly, $\phi' \circ \phi' = \phi'$ and $\phi' \mid_{P \oplus 1 \oplus \cdots \oplus 1} = id$.

$$\begin{aligned} \phi' \circ \phi'((p, 1, \cdots, 1)) &= \phi'(p, 1, \cdots, 1) \\ \phi'((p, 1, \cdots, 1)) &= (p, 1, \cdots, 1). \end{aligned}$$

Hence, $P \oplus 1 \oplus \cdots \oplus 1$ is a retract of $P \oplus Q_1 \oplus \cdots \oplus Q_n$, so P is a retract of $\mathcal{F}$. This is *property 3 in Question 2*, and it implies P is projective. Therefore we've shown that the if direction.

Class Notes, Exercises 3.21

Question 1 In this problem, we will show $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as a $\mathbb{Z}$-module.

a. Show that $\varphi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ given by $\varphi(r \otimes_{\mathbb{Z}} s) = rs$ for r and $s \in \mathbb{Q}$ is a homomorphism. To do this we need to find the appropriate middle linear map.

Answer: Let's define a map $\iota : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\iota(r, s) = r \otimes_{\mathbb{Z}} s$. This map is a middle linear map. It directly follows the definition of the tensor product.

$$\begin{aligned} &\iota(rn, s) = rn \otimes_{\mathbb{Z}} s = r \otimes_{\mathbb{Z}} ns = \iota(r, ns) \\ &\iota(r_1 + r_2, s) = (r_1 + r_2) \otimes_{\mathbb{Z}} s = r_1 \otimes_{\mathbb{Z}} s + r_2 \otimes_{\mathbb{Z}} s = \iota(r_1, s) + \iota(r_2, s) \\ &\iota(r, s_1 + s_2) = r \otimes_{\mathbb{Z}} (s_1 + s_2) = r \otimes_{\mathbb{Z}} s_1 + r \otimes_{\mathbb{Z}} s_2 = \iota(r, s_1) + \iota(r, s_2). \end{aligned}$$

Let's also define $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ by $\phi(r, s) = rs$. We can show this map is also a middle linear map.

$$\begin{aligned}
&\phi(rn, s) = rns = \phi(r, ns) \\
&\phi(r_1 + r_2, s) = (r_1 + r_2) \cdot s = r_1 \cdot s + r_2 \cdot s = \phi(r_1, s) + \phi(r_2, s) \\
&\phi(r, s_1 + s_2) = r \cdot (s_1 + s_2) = r \cdot s_1 + r \cdot s_2 = \phi(r, s_1) + \phi(r, s_2).
\end{aligned}$$

According to *Theorem 10* on Dummit & Foot, if R is a ring with 1, M is a right R-module, N is a left R-module, L is an abelian group, and $\alpha : M \times N \to M \otimes_R N$, $\beta : M \times N \to L$ are middle linear maps, then there exist a unique homomorphism $\gamma : M \otimes_R N \to L$, such that $\gamma \circ \alpha = \beta$. The following diagram commutes.

$$\begin{array}{ccc}
M \times N & \overset{\alpha}{\to} & M \otimes_R N \\
& {}^{\beta}\searrow & \downarrow^{\gamma} \\
& & L
\end{array}$$

$\mathbb{Z}$ is a ring with 1, $\mathbb{Q}$ is and a left and a right $\mathbb{Z}$-module, and $\mathbb{Q}$ is an abelian group. And we've shown $\iota : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ are middle linear maps. Also, $\varphi \circ \iota(r, s) = \varphi(r \otimes_{\mathbb{Z}} s) = rs = \phi(r, s)$. So we conclude that φ defined in this problem is the unique homomorphism from $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathbb{Q}$

$$\begin{array}{ccc}
\mathbb{Q} \times \mathbb{Q} & \overset{\iota}{\to} & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \\
& {}^{\phi}\searrow & \downarrow^{\varphi} \\
& & \mathbb{Q}
\end{array}$$

b. Show that if r and $s \in \mathbb{Q}$ then $r \otimes_{\mathbb{Z}} s = 1 \otimes_{\mathbb{Z}} rs$.

Answer: According to the definition of tensor product, we have $mr \otimes_R n = m \otimes_R rn$ where $r \in R$.

$$\begin{aligned}
& r \otimes_{\mathbb{Z}} s = \left(\frac{r_1}{r_2}\right) \otimes_{\mathbb{Z}} s, \text{ where, } r_1, r_2 \in \mathbb{Z} \\
\Rightarrow \quad & r \otimes_{\mathbb{Z}} s = \left(\frac{r_1}{r_2}\right) \otimes_{\mathbb{Z}} \left(\frac{r_2}{r_2} \cdot s\right) = \left(\frac{r_2}{r_2}\right) \otimes_{\mathbb{Z}} \left(\frac{r_1}{r_2} \cdot s\right) = 1 \otimes_{\mathbb{Z}} rs.
\end{aligned}$$

c. Show that $r \mapsto 1 \otimes_{\mathbb{Z}} r$ is a homomorphism and is the inverse of φ.

Answer: To show the given map, $\varphi' : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, with $\varphi' : r \mapsto 1 \otimes_{\mathbb{Z}} r$, is a homomorphism, we need to show $\varphi'(r + s) = \varphi'(r) + \varphi'(s)$ and $\varphi'(rs) = \varphi'(r)\varphi'(s)$ for $\forall r, s \in \mathbb{Q}$.

$$\begin{aligned}
&\varphi'(r + s) = 1 \otimes_{\mathbb{Z}} (r + s) = 1 \otimes_{\mathbb{Z}} r + 1 \otimes_{\mathbb{Z}} s = \varphi'(r) + \varphi'(s) \\
&\varphi'(rs) = 1 \otimes_{\mathbb{Z}} rs = (1 \otimes_{\mathbb{Z}} r)(1 \otimes_{\mathbb{Z}} s) = \varphi'(r)\varphi'(s).
\end{aligned}$$

Now we show that φ' is the inverse of φ.

$$\begin{aligned} & \varphi \circ \varphi'(r) = \varphi(1 \otimes_{\mathbb{Z}} r) = r \\ & \varphi' \circ \varphi(r \otimes_{\mathbb{Z}} s) = \varphi'(rs) = 1 \otimes_{\mathbb{Z}} rs = r \otimes_{\mathbb{Z}} s \\ \Rightarrow \quad & \varphi \circ \varphi' = \varphi' \circ \varphi = id_{\mathbb{Q}}. \end{aligned}$$

At this point, we've shown, there is surjective homomorphism $\varphi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, a surjective homomorphism $\varphi' : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ since $\forall r \otimes_{\mathbb{Z}} s \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\exists rs \in \mathbb{Q}$, with $\varphi'(rs) = 1 \otimes_{\mathbb{Z}} rs = r \otimes_{\mathbb{Z}} s$, and $\varphi \circ \varphi' = id_{\mathbb{Q}}$. So, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

Question 2 Suppose R is commutative and I and J are ideals of R. Show that

$$R/I \otimes R/J \cong R/(I \vee J).$$

$I \vee J$ is the ideal generated by I and J. It is often written $I + J$. For the map $R/I \otimes R/J \to R/(I \vee J)$ we need to make a middle linear map $R/I \times R/J \to R/(I \vee J)$. For the other direction map $R \to R/I \otimes R/J$ by $r \mapsto r(1 \otimes 1)$ and show that $I \vee J$ is contained in the kernel.

Answer: Let's construct a map, $\alpha : R/I \times R/J \to R/I \otimes R/J$, defining $\alpha((r_1 + I), (r_2 + J)) = (r_1 + I) \otimes (r_2 + J)$. This map is a middle linear map. This is directly following the definition of the tensor product.

$$\begin{aligned} \alpha((r_1 + I) \cdot r, (r_2 + J)) &= (r_1 + I) \cdot r \otimes (r_2 + J) \\ &= (r_1 + I) \otimes r \cdot (r_2 + J) \\ &= \alpha((r_1 + I), r \cdot (r_2 + J)). \\ \alpha(((r_1 + r_2) + I), (r_3 + J)) &= ((r_1 + r_2) + I) \otimes (r_3 + J) \\ &= ((r_1 + I) + (r_2 + I)) \otimes (r_3 + J) \\ &= (r_1 + I) \otimes (r_3 + J) + (r_2 + I) \otimes (r_3 + J) \\ &= \alpha((r_1 + I), (r_3 + I)) + \alpha((r_2 + I), (r_3 + I)) \\ \alpha((r_1 + I), ((r_2 + r_3) + J)) &= (r_1 + I) \otimes ((r_2 + r_3) + J) \\ &= (r_1 + I) \otimes ((r_2 + J) + (r_3 + J)) \\ &= (r_1 + I) \otimes (r_2 + J) + (r_1 + I) \otimes (r_3 + J) \\ &= \alpha((r_1 + I), (r_2 + I)) + \alpha((r_1 + I), (r_3 + I)). \end{aligned}$$

Let's also construct a map, $\beta : R/I \times R/J \to R/(I \vee J)$, defining $\beta((r_1 + I), (r_2 + J)) =$

$r_1 r_2 + I \vee J$. We can show this map is also a middle linear map.

$$\begin{aligned}
\beta\left(\left(r_1+I\right)r,\left(r_2+J\right)\right) &= \beta\left(\left(r_1 r+I\right),\left(r_2+J\right)\right)\\
&= r_1 r r_2 + I \vee J\\
&= \beta\left(\left(r_1+I\right),\left(r_2 r+J\right)\right)\\
&= \beta\left(\left(r_1+I\right),r\left(r_2+J\right)\right).\\
\beta\left(\left(\left(r_1+r_2\right)+I\right),\left(r_3+J\right)\right) &= \left(r_1+r_2\right)r_3\\
&= r_1 r_3 + r_2 r_3\\
&= \beta\left(\left(r_1+I\right),\left(r_3+J\right)\right)+\beta\left(\left(r_2+I\right),\left(r_3+J\right)\right)\\
\beta\left(\left(r_1+J\right),\left(\left(r_2+r_3\right)+I\right)\right) &= r_1\left(r_2+r_3\right)\\
&= r_1 r_2 + r_1 r_3\\
&= \beta\left(\left(r_1+I\right),\left(r_2+J\right)\right)+\beta\left(\left(r_1+I\right),\left(r_3+J\right)\right).
\end{aligned}$$

According to *Theorem 10* in Dummit & Foote, there exist a homomorphism $\phi : R/I \otimes R/J \to R/\left(I \vee J\right)$, such that $\beta = \phi \circ \alpha$. The following diagram commutes.

$$\begin{array}{ccc}
R/I \times R/J & \overset{\alpha}{\to} & R/I \otimes_R R/J \\
& {}^{\beta}\searrow & \downarrow^{\phi} \\
& & R/\left(I \vee J\right)
\end{array}$$

We can derive the homomorphism ϕ and show that ϕ is surjective.

$$\begin{aligned}
\beta\left(\left(r_1+I\right),\left(r_2+J\right)\right) &= r_1 r_2 + I \vee J\\
\alpha\left(\left(r_1+I\right),\left(r_2+J\right)\right) &= \left(r_1+I\right)\otimes\left(r_2+J\right)\\
\Rightarrow \phi : \left(r_1+I\right)\otimes\left(r_2+J\right) &\mapsto r_1 r_2 + I \vee J\\
\Rightarrow \forall\left(r + I \vee J\right) \in R/\left(I \vee J\right) &\;\exists\; \left(1+I\right)\otimes\left(r+J\right) \in R/I \otimes_R R/J\\
\phi\left(\left(1+I\right)\otimes\left(r+J\right)\right) &= r + I \vee J.
\end{aligned}$$

For the other direction, let's construct a map $\varphi : R \to R/I \otimes R/J$ by defining $\varphi : r \mapsto r\left(1_{R/I} \otimes 1_{R/J}\right)$. This is a homomorphism.

$$\begin{aligned}
\varphi\left(r_1\right)+\varphi\left(r_2\right) &= r_1\left(\left(1+I\right)\otimes\left(1+J\right)\right)+r_2\left(\left(1+I\right)\otimes\left(1+J\right)\right)\\
&= \left(r_1+r_2\right)\left(\left(1+I\right)\otimes\left(1+J\right)\right)\\
&= \varphi\left(r_1+r_2\right)\\
\varphi\left(r_1\right)\varphi\left(r_2\right) &= \left(\left(r_1+I\right)\otimes\left(1+J\right)\right)\left(\left(r_2+I\right)\otimes\left(1+J\right)\right)\\
&= \left(\left(r_1+I\right)\left(r_2+I\right)\otimes\left(1+J\right)\left(1+J\right)\right)\\
&= r_1 r_2\left(\left(1+I\right)\otimes\left(1+J\right)\right)\\
&= \varphi\left(r_1 r_2\right).
\end{aligned}$$

We can show that $I \vee J$ is in the kernel of φ.

$$\begin{aligned}
\forall i+j \in I \vee J & \qquad \text{where } i \in I\ j \in J \\
\Rightarrow \varphi(i+j) &= (i+j)\left((1+I)\otimes(1+J)\right) \\
&= i\left((1+I)\otimes(1+J)\right) + j\left((1+I)\otimes(1+J)\right) \\
&= \left((i+I)\otimes(1+J)\right) + \left((1+I)\otimes(j+J)\right) \\
&= \left(I\otimes(1+J)\right) + \left((1+I)\otimes J\right) \\
&= 0_{R/I\otimes R/J} + 0_{R/I\otimes R/J} \\
&= 0_{R/I\otimes R/J}.
\end{aligned}$$

So we have a homomorphism $\varphi' : R/(I\vee J) \to R/I\otimes R/J$ with $\varphi' : r+I\vee J \mapsto r\left(1_{R/I}\otimes 1_{R/J}\right)$. We can show that φ' is surjective. In other words, any tensor in $R/I\otimes R/J$ can be written as a scalar multiple of $1_{R/I}\otimes 1_{R/J}$.

$$\begin{aligned}
(r_1+I)\otimes(r_2+J) &= \left(r_1(1+I)\right)\otimes\left(r_2(1+J)\right) \\
&= r_1r_2\left((1+I)\otimes(1+J)\right) \\
&= \varphi'\left(r_1r_2+I\vee J\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \forall (r_1+I)\otimes(r_2+J) &\in R/I\otimes R/J \\
\exists (r_1r_2+I\vee J)\in R/(I\vee J) \quad &\text{with} \quad \varphi'(r_1r_2+I\vee J) = (r_1+I)\otimes(r_2+J).
\end{aligned}$$

In fact ϕ and φ' are inverses.

$$\begin{aligned}
\phi\left(\varphi'(r+I\vee J)\right) &= \phi\left(r\left((1+I)\otimes(1+J)\right)\right) \\
&= \phi\left((r+I)\otimes(1+J)\right) \\
&= r+I\vee J \\
&\Rightarrow \phi\circ\varphi' = id_{R/(I\vee J)} \\
\varphi'\left(\phi\left((r_1+I)\otimes(r_2+J)\right)\right) &= \varphi'(r_1r_2+I\vee J) \\
&= (r_1+I)\otimes(r_2+J) \\
&\Rightarrow \varphi'\circ\phi = id_{R/I\otimes R/J}.
\end{aligned}$$

We've shown there exist two inverse surjective homomorphisms between $R/(I\vee J)$ and $R/I\otimes R/J$, so $R/I\otimes R/J \cong R/(I\vee J)$.

Question 3 Using the previous problem and the fundamental theorem of abelian groups and that tensor products distribute over direct sum, describe $A\otimes_{\mathbb{Z}} B$, where A and B are finite abelian groups. Alternatively, find $A\otimes_{\mathbb{Z}} B$, where $A = \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/16\mathbb{Z}$ and $B = \mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/27\mathbb{Z}$.

Answer: The *fundamental theorem of finite abelian groups* states that every finite abelian group G can be expressed as the direct sum of cyclic subgroups of prime-power order. A and B are already expressed as the direct sum of cyclic subgroups of prime-power order.

$$A \otimes_{\mathbb{Z}} B \quad = \quad (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}) \, .$$

According to *Theorem 17* on Dummit & Foote, tensor product distribute over direct sums. If $M, \ M'$ are right R-modules and $N, \ N'$ are left R-modules, then there are group isomorphisms, $(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$ and $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$. So we have the following conversion.

$$\begin{aligned} A \otimes_{\mathbb{Z}} B &= (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}) \\ &= (\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/27\mathbb{Z}) \oplus (\mathbb{Z}/16\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/16\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/27\mathbb{Z}) \, . \end{aligned}$$

We've shown in the previous problem that $R/I \otimes R/J \cong R/\left(I \vee J\right)$ where R is commutative, $I, \ J$ are ideals of R. For $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/b\mathbb{Z}$, we have $R = \mathbb{Z}$, which is a commutative ring, $I = a\mathbb{Z} = \{i \in \mathbb{Z} : i = a \cdot n, \ n = 0, 1, 2, \cdots\}$, $J = b\mathbb{Z} = \{j \in \mathbb{Z} : j = b \cdot m, \ m = 0, 1, 2, \cdots\}$. We can show that $I \vee J$ here is the ideal generated by the greatest common denominator of a and b.

$$\begin{aligned} a\mathbb{Z} \vee b\mathbb{Z} &= \{x : x = i + j, \ i \in I, \ j \in J\} \\ \Rightarrow a\mathbb{Z} \vee b\mathbb{Z} &= \{x : x = a \cdot n + b \cdot m, \ m, n \in \mathbb{Z}\} \\ \because &\quad \mathbb{Z} \text{ is a PID.} \\ \therefore &\quad a\mathbb{Z} = (a) \, , \ b\mathbb{Z} = (b) \, , \ a\mathbb{Z} \vee b\mathbb{Z} = (gcd\,(a, b)) \, . \end{aligned}$$

According to the conclusion from the previous problem we have the following relations.

$$\begin{aligned} &\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z} \cong \mathbb{Z}/\left(gcd\,(4, 8)\right) = \mathbb{Z}/4\mathbb{Z} \\ &\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/27\mathbb{Z} \cong \mathbb{Z}/\left(gcd\,(4, 27)\right) = \mathbb{Z}/\mathbb{Z} = \{0\} \\ &\mathbb{Z}/16\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z} \cong \mathbb{Z}/\left(gcd\,(16, 8)\right) = \mathbb{Z}/8\mathbb{Z} \\ &\mathbb{Z}/16\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/27\mathbb{Z} \cong \mathbb{Z}/\left(gcd\,(16, 27)\right) = \mathbb{Z}/\mathbb{Z} = \{0\} \end{aligned}$$

$$\begin{aligned} \Rightarrow A \otimes_{\mathbb{Z}} B &= (\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/27\mathbb{Z}) \oplus (\mathbb{Z}/16\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/16\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/27\mathbb{Z}) \\ &= \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}. \end{aligned}$$

Question 4 Let B_1 and B_2 be submodules of the left R-module A. Let D be a flat right R-module. Show,

$$\begin{aligned} D \otimes_R (B_1 \vee B_2) &= (D \otimes_R B_1) \vee (D \otimes_R B_2) \\ D \otimes_R (B_1 \cap B_2) &= (D \otimes_R B_1) \cap (D \otimes_R B_2) \,. \end{aligned}$$

proving the map $B \mapsto D \otimes_R B$ is a lattice homomorphism of $\mathbf{Sub}\,(A) \to \mathbf{Sub}\,(D \otimes_R A)$. Hint, the hardest part is proving the inclusion $(D \otimes_R B_1) \cap (D \otimes_R B_2) \subseteq D \otimes_R (B_1 \cap B_2)$. To see this first note $B_1 / (B_1 \cap B_2) \cong (B_1 \vee B_2) / B_2$. Hence we have the short exact sequence,

$$0 \to B_1 \cap B_2 \to B_1 \xrightarrow{\varphi} (B_1 \vee B_2) / B_2 \to 0.$$

and hence the following sequence is exact, so $ker\,(1 \otimes_R \varphi) = D \otimes_R (B_1 \cap B_2)$. Use this to show $(D \otimes_R B_1) \cap (D \otimes_R B_2) \subseteq D \otimes_R (B_1 \cap B_2)$.

$$0 \to D \otimes_R B_1 \cap B_2 \to D \otimes_R B_1 \xrightarrow{1 \otimes_R \varphi} D \otimes_R (B_1 \vee B_2) / B_2 \to 0.$$

Answer: *<u>First</u>*, we will prove $D \otimes_R (B_1 \vee B_2) = (D \otimes_R B_1) \vee (D \otimes_R B_2)$. We construct the following sequence, and define $id : B_1 \to B_1 \vee B_2$ by $id : b_1 \mapsto b_1$.

$$0 \to B_1 \xrightarrow{id} B_1 \vee B_2 \xrightarrow{\alpha} (B_1 \vee B_2) / B_1 \to 0.$$

We know that $0 \to ker\,(\phi) \to X \xrightarrow{\phi} img\,(\phi) \to 0$, where ϕ is a homomorphism, always forms a short exact sequence. The the sequence above, $0 \to B_1 \xrightarrow{id} B_1 \vee B_2 \xrightarrow{\alpha} (B_1 \vee B_2) / B_1 \to 0$ is a short exact sequence. D is flat, so we have the following short exact sequence.

$$0 \to D \otimes_R B_1 \xrightarrow{1 \otimes_R id} D \otimes_R (B_1 \vee B_2) \xrightarrow{1 \otimes_R \alpha} D \otimes_R ((B_1 \vee B_2) / B_1) \to 0.$$

Sine the map $1 \otimes_R id$ is just the identity map, and it is injective, we have $D \otimes_R B_1$ is a submodule contained in $D \otimes_R (B_1 \vee B_2)$. So we have the following relations and we can derive $(D \otimes_R B_1) \vee (D \otimes_R B_2) \leq D \otimes_R (B_1 \vee B_2)$.

$$\begin{aligned} & D \otimes_R B_1 \leq D \otimes_R (B_1 \vee B_2) \\ & D \otimes_R B_2 \leq D \otimes_R (B_1 \vee B_2) \\ \Rightarrow\quad & d_1 \otimes_R b_1 + d_2 \otimes_R b_2 \in D \otimes_R (B_1 \vee B_2)\,, \\ & \text{where } (d_1 \otimes_R b_1 + d_2 \otimes_R b_2) \text{ is a typical element in } (D \otimes_R B_1) \vee (D \otimes_R B_2) \\ \Rightarrow\quad & (D \otimes_R B_1) \vee (D \otimes_R B_2) \leq D \otimes_R (B_1 \vee B_2)\,. \end{aligned}$$

Now we've shown one direction. we can show the other direction by evaluating a typical element from $D \otimes_R (B_1 \vee B_2)$, $d \otimes_R (b_1 + b_2)$ where $d \in D$, $b_1 \in B_1$, $b_2 \in B_2$. According to the properties of tensor products, we have the following relation.

$$\begin{aligned} d \otimes_R (b_1 + b_2) &= d \otimes_R b_1 + d \otimes_R b_2 \\ &\in (D \otimes_R B_1) \vee (D \otimes_R B_2) \\ &\Rightarrow D \otimes_R (B_1 \vee B_2) \leq (D \otimes_R B_1) \vee (D \otimes_R B_1) . \end{aligned}$$

Combining these two conclusions, we have $(D \otimes_R B_1) \vee (D \otimes_R B_1) = D \otimes_R (B_1 \vee B_2)$.

<u>Second</u>, we will prove $D \otimes_R (B_1 \cap B_2) = (D \otimes_R B_1) \cap (D \otimes_R B_2)$. We can easily show $D \otimes_R (B_1 \cap B_2) \leq (D \otimes_R B_1) \cap (D \otimes_R B_2)$ by evaluating a typical element in $D \otimes_R (B_1 \cap B_2)$, $d \otimes_R b$ where $d \in D$, $b \in B_1$, $b \in B_2$

$$\begin{aligned} d \otimes_R b &\in D \otimes_R B_1 \ \because b \in B_1 \\ d \otimes_R b &\in D \otimes_R B_2 \ \because b \in B_2 \\ \Rightarrow d \otimes_R b &\in (D \otimes_R B_1) \cap (D \otimes_R B_2) \\ \Rightarrow D \otimes_R (B_1 \cap B_2) &\leq (D \otimes_R B_1) \cap (D \otimes_R B_2) . \end{aligned}$$

To show the other direction, we first construct the following sequences. They are short exact sequences, since $0 \to ker(\phi) \to X \stackrel{\phi}{\to} img(\phi) \to 0$, where ϕ is a homomorphism, always forms a short exact sequence.

$$\begin{gathered} 0 \to B_1 \cap B_2 \stackrel{id_{B_1}}{\to} B_1 \stackrel{\beta_1}{\to} B_1 / (B_1 \cap B_2) \to 0 \\ 0 \to B_1 \cap B_2 \stackrel{id_{B_2}}{\to} B_2 \stackrel{\beta_2}{\to} B_2 / (B_1 \cap B_2) \to 0 \end{gathered}$$

According to the second isomorphism theorem, $B_1 / (B_1 \cap B_2) \cong (B_1 \vee B_2) / B_2$. So, we convert the sequences above.

$$\begin{gathered} 0 \to B_1 \cap B_2 \stackrel{id_{B_1}}{\to} B_1 \stackrel{\beta_1'}{\to} (B_1 \vee B_2) / B_2 \to 0 \\ 0 \to B_1 \cap B_2 \stackrel{id_{B_2}}{\to} B_2 \stackrel{\beta_2'}{\to} (B_1 \vee B_2) / B_1 \to 0 \end{gathered}$$

Since D is flat, we have the following short exact sequences.

$$\begin{gathered} 0 \to D \otimes_R (B_1 \cap B_2) \stackrel{1 \otimes_R id_{B_1}}{\to} D \otimes_R B_1 \stackrel{1 \otimes_R \beta_1'}{\to} D \otimes_R ((B_1 \vee B_2) / B_2) \to 0 \\ 0 \to D \otimes_R (B_1 \cap B_2) \stackrel{1 \otimes_R id_{B_2}}{\to} D \otimes_R B_2 \stackrel{1 \otimes_R \beta_2'}{\to} D \otimes_R ((B_1 \vee B_2) / B_1) \to 0. \end{gathered}$$

According to the definition of the short exact sequence, $img(1 \otimes_R id) = ker(1 \otimes_R \beta')$ for both sequences.

$$\begin{aligned}
& 1 \otimes_R id_{B_1} : D \otimes_R (B_1 \cap B_2) \to D \otimes_R B_1 \\
& 1 \otimes_R id_{B_2} : D \otimes_R (B_1 \cap B_2) \to D \otimes_R B_2 \\
\Rightarrow \quad & img\,(1 \otimes_R id_{B_1}) = img\,(1 \otimes_R id_{B_2}) = D \otimes_R (B_1 \cap B_2) \\
\Rightarrow \quad & ker\,(1 \otimes_R \beta') = D \otimes_R (B_1 \cap B_2)\,.
\end{aligned}$$

Sine everything in $D \otimes_R B_1$ maps to the kernel by $1 \otimes_R \beta_2' : D \otimes_R B_2 \to D \otimes_R ((B_1 \vee B_2)/B_1)$, with $1 \otimes_R \beta_2' : d \otimes_R b_2 \mapsto d \otimes_R (b_2 + B_1)$, we have the following relation.

$$(D \otimes_R B_2) \cap (D \otimes_R B_1) \subseteq ker\,(1 \otimes_R \beta_2')\,.$$

Sine everything in $D \otimes_R B_2$ maps to the kernel by $1 \otimes_R \beta_1' : D \otimes_R B_1 \to D \otimes_R ((B_1 \vee B_2)/B_2)$, with $1 \otimes_R \beta_1' : d \otimes_R b_1 \mapsto d \otimes_R (b_1 + B_2)$, we have the following relation.

$$(D \otimes_R B_1) \cap (D \otimes_R B_2) \subseteq ker\,(1 \otimes_R \beta_1')\,.$$

Together, we've shown that $(D \otimes_R B_1) \cap (D \otimes_R B_2)$ is a subset of $D \otimes_R (B_1 \cap B_2)$.

$$\begin{aligned}
& (D \otimes_R B_1) \cap (D \otimes_R B_2) \subseteq ker\,(1 \otimes_R \beta_1') = D \otimes_R (B_1 \cap B_2) \\
& \Rightarrow (D \otimes_R B_1) \cap (D \otimes_R B_2) \subseteq D \otimes_R (B_1 \cap B_2)\,.
\end{aligned}$$

Combining with the earlier results, we conclude that $(D \otimes_R B_1) \cap (D \otimes_R B_2) = D \otimes_R (B_1 \cap B_2)\,.$

$$\begin{aligned}
\because \quad & D \otimes_R (B_1 \cap B_2) \leq (D \otimes_R B_1) \cap (D \otimes_R B_2) \\
& (D \otimes_R B_1) \cap (D \otimes_R B_2) \subseteq D \otimes_R (B_1 \cap B_2) \\
\Rightarrow \quad & D \otimes_R (B_1 \cap B_2) = (D \otimes_R B_1) \cap (D \otimes_R B_2)\,.
\end{aligned}$$