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ICS 241

Recitation Lecture Note #1

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Question: Show that the program segment S is correct with respect to the initial assertion $p: b = 3$ and the final assertion $q: c = 11$. [Chapter 4.5 Review]

$a := 5$
$c := a + 2b$

Answer: Suppose p is true. Therefore $b = 3$ at the beginning of the program. As the program runs, 5 is assigned to a and then $5 + 2 \cdot 3$, or 11, is assigned to c. Therefore, $p\{S\}q$ is true.

Question: Verify that the program segment [Chapter 4.5 Review]

```
x := 0
z := x + y
if y > 0 then
        z := z + 1
else
        z := 0
```

Is correct with respect to the initial assertion $\mathrm{y} = 3$ and the final assertion $\mathrm{z} = 6$.

Answer: Suppose that $y = 3$. The program segment assigns the value 2 to x and then assigns the value $x + y = 2 + 3 = 5$ to z. Because $y = 3 > 0$ it then assigns the value $z + 1 = 5 + 2 = 6$ to z.

Question: Use a loop invariant to prove that this program segment for computing nx (x a real number), where n is a positive integer, is correct: [Chapter 4.5 Review]

```
multiple := 0
i := 1
while i ≤ n
begin
        multiple := multiple + x
        i := i + 1
end
```

Answer: We will show that the following is a loop invariant.

$$p: multiple = (i - 1)x \text{ and } i \leq n + 1$$

Initially p is true because $i = 1$ and $multiple = 0 = (1 - 1)x$. Now suppose that p is true and $i \leq n$ after the loop is executed. We must show that p is true after another execution of the loop. Because $i \leq n$, after one more execution of the loop, i will be incremented by 1 and we have $i \leq n + 1$. Also, $multiple$ becomes $multiple + x$, or $(i - 1)x + x = ix$. Hence p remains true. Therefore, p is a loop invariant. Finally, the loop terminates with $i = n + 1$ after n traversals of the loop because $i = 1$ prior to the loop and each traversal of the loop adds 1 to n. Thus, at termination $multiple = nx$.

Question: Give a recurrence relation and initial conditions for the number of n-digit decimal strings containing an even number of 0 digits. [Chapter 7.1 Review]

Answer: *Base case:* $a_0 = 0$, $a_1 = 9$.

Recursion: for $n > 1, n - 1$ digits, we have a number of a_{n-1} valid strings. Meanwhile, we have $10^{n-1} - a_{n-1}$ invalid strings.

The number of valid strings for n digits is made up of two sub-cases:

$9 \cdot a_{n-1}$	Any valid string append with a digit from $1 - 9$.
$10^{n-1} - a_{n-1}$	Any invalid string append with a digit 0

Therefore, we have $a_n = 9 \cdot a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$

Double check: $a_2 = 9 \cdot 9 + 1 = 82$

$8 \cdot a_1 + 10^0 = 8 \cdot 9 + 1 = 82$ check!

$a_3 = 9 \cdot 9 \cdot 9 + 9 \cdot 3 = 756$

$8 \cdot a_2 + 10^1 = 8 \cdot 82 + 10 = 756$ check!

Question: Find the recurrence relation (and initial condition) for each of the following: [Chapter 7.1 Review]

a. the number of strings of length n of letter of the alphabet.

Answer: Let an equal the number of strings of length n of letters of the alphabet. We can obtain any such string by taking a string s of length $n - 1$ and appending a letter to the end of s. This can be done in 26 ways. Therefore, $a_n = 26a_{n-1}$. The initial condition is $a_1 = 26$.

b. the number of strings of length n of letter of the alphabet, if no adjacent letters can be the same.

Answer: Let b_n equal the number of strings of length n of letters of the alphabet with no adjacent letters identical. Each such string can be obtained from a string s of length $n - 1$ by taking s

and appending to it a letter that is different from the last letter of s. Because there are 25 letters that can be appended, there are 25 ways to extend s to a string of length n. Therefore, $b_n = 25b_{n-1}$. The initial condition is $b_1 = 26$.

c. the number of strings of length n of letter of the alphabet with no repeated letters.

Answer: Let c_n equal the number of strings of length n of letters of the alphabet with no repeated letters. Each such string can be obtained from a string s of length $n - 1$ by taking s and appending to it a letter that is different from each of the letters of s. Because there are $n - 1$ letters used in s, there are $26 - (n - 1) = 27 - n$ letters available to be appended to s. Therefore, $c_n = (27 - n)c_{n-1}$. The initial condition is $c_1 = 26$. Note that $c_{27} = (27 - 27)c_{26} = 0$ because there are only 26 letters in the alphabet. Likewise, the recurrence relation yields $c_{28} = c_{29} = \ldots = 0$.

Question: Find a recurrence relation for the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$, which is given by the formula $a_n = \frac{1}{2n+1}$ for $n = 0, 1, 2, 3, \cdots$. [Chapter7.1 Review]

Answer: We will try to relate $a_n = \frac{1}{2n+1}$ and $a_{n-1} = \frac{1}{2(n-1)+1} = \frac{1}{2n-1}$ to each other. We can rewrite $\frac{1}{a_{n-1}} = 2n - 1$.

$$a_n = \frac{1}{2n+1} = \frac{1}{2n-1+2} = \frac{1}{\frac{1}{a_{n-1}} + 2} = \frac{1}{\frac{1+2a_{n-1}}{a_{n-1}}} = \frac{a_{n-1}}{1+2a_{n-1}}$$

Thus a recurrence relation for the given sequences is

$$a_n = \frac{a_{n-1}}{1+2a_{n-1}}$$

with initial condition $a_0 = 1$.

Alternatively, we could also do:

$$\frac{a_n}{a_{n-1}} = \frac{\frac{1}{2n+1}}{\frac{1}{2n-1}} = \frac{2n-1}{2n+1}$$

Therefore:

$$a_n = \frac{2n-1}{2n+1} \cdot a_{n-1}$$

with initial condition $a_0 = 1$.

Question: (Problem A1 from the 1990 William Lowell Putnam Mathematics Competition)

Here are the first ten terms of an infinite sequence:

$$2, 3, 6, 14, 40, 152, 784, 5168, 40567, 363392.$$

a. Find a formula for an infinite sequence $a_0, a_1, a_2, a_3, \ldots$ such that the first ten terms of the sequence are the ones given here. (Hint: consider the sum of two rapidly increasing sequences.)

Answer: $a_n = 2^n + n!$

b. Show that the sequence in **a** satisfies the recurrence relation

$$a_n = (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3}$$

Answer: $a_n = (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3}$

$$a_n = (n+4)(2^{n-1} + (n-1)!) - 4n(2^{n-2} + (n-2)!) + (4n-8)(2^{n-3} + (n-3)!)$$

$$a_n = (n+4)(2^{n-1} + (n-1)!) - 4n(2^{n-2} + (n-2)!) + (n-2)2^{n-1} + 4(n-2)!$$

$$a_n = (n+4)(2^{n-1} + (n-1)!) - n2^n - 4n(n-2)! + (n-2)2^{n-1} + 4(n-2)!$$

$$a_n = n2^{n-1} + 2^{n+1} + 4(n-1)(n-2)! + n! - n2^n - 4n(n-2)! + (n-2)2^{n-1} + 4(n-2)!$$

$$a_n = n2^{n-1} + 2^{n+1} + n! - n2^n + (n-2)2^{n-1}$$

$$a_n = n2^{n-1} + 2^{n+1} + n! - 2n2^{n-1} + n2^{n-1} - 2^n$$

$$a_n = 2^{n+1} + n! - 2^n$$

$$a_n = 2^n + n!$$

Question: Determine the following equations satisfy associated homogeneous recurrence relation, or linear nonhomogeneous recurrence relation with constant coefficients. [Chapter 7.2 Review]

Answer:

$a_n = -3a_{n-1} + a_{n-2} + 2a_{n-3}$	homogeneous
$a_n = -3a_{n-1}$	homogeneous
$a_n = 2a_{n-1} + a_{n-2} + n^2 + n + 1$	nonhomogeneous
$a_n = a_{n-1} + 2^n$	nonhomogeneous

Question: Write the set {2,3,4} (given in list notation) in set builder notation. [Chapter 2.1 Review]

Answer: Here are three ways of writing the set in set builder notation:

$$\{\, x \mid x \in N, 1 < x < 5 \,\},$$

$\{ x \mid x \in N, 2 \leq x \leq 4 \}$,

$\{ x \mid x \in R, x3 - 9x2 + 26x - 24 = 0 \}$.

This last set was obtained by taking the equation $(x-2)(x-3)(x-4) = 0$ and multiplying out the left side.

Question: Use logical equivalence to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. [Chapter 2.2 Review]

Answer:

$A \cap (B \cup C) = \{x \mid x \in A \land x \in B \cup C\}$	Definition of intersection
$= \{x \mid x \in A \land (x \in B \lor x \in C)\}$	Definition of union
$= \{x \mid (x \in A \land x \in B) \lor (x \in A \land x \in C)\}$	Distributive law
$= \{x \mid (x \in A \cap B) \lor (x \in A \cap C)\}$	Definition of intersection
$= (A \cap B) \cup (A \cap C)$	Definition of union

Question: Express in sigma notation the sum of the first 50 terms of the series $3 + 6 + 9 + 12 + 15 + \cdots$ [Chapter 2.4 Review]

Answer: In sigma notation we have $\sum_{i=1}^{50} 3i$. Note that we could also write this in other forms, for example $\sum_{j=1}^{50} 3j$, $\sum_{k=1}^{50} 3k$ (we can use any variable as the index of summation). We can also change the limits of summation, obtaining forms such as the sum $\sum_{i=0}^{49} 3(i+1)$.

Question: Find the prime factorization of: [Chapter 3.5 Review]

a. 487

Answer: we try to divide 487 by all primes from 2 to $\lfloor\sqrt{487}\rfloor = 22$ (that is, $2, 3, 5, 7, 11, 13, 17, 19$), we find that none of these divides 487 without a remainder. Therefore 487 is prime.

b. 6600

Answer: Begin by writing 6600 as any product of smaller positive factors, such as $6600 = 66 \cdot 100$. We continue this process until only primes are obtained:

$$\begin{aligned} 6600 &= 66 \cdot 100 \\ &= (6 \cdot 11) \cdot (10 \cdot 10) \\ &= (3 \cdot 3 \cdot 11) \cdot (2 \cdot 5 \cdot 2 \cdot 5) \\ &= 2^3 \cdot 3 \cdot 5^2 \cdot 11 \end{aligned}$$

Question: Let $A = \begin{bmatrix} 2 & 7 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$. Find the products AB and BA. [Chapter 3.8 Review]

Answer:

$$AB = \begin{bmatrix} 2 & 7 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 7\cdot 4 & 2\cdot(-3) + 7\cdot 2 \\ (-1)\cdot 1 + 5\cdot 4 & (-1)\cdot(-3) + 5\cdot 2 \end{bmatrix} = \begin{bmatrix} 30 & 8 \\ 19 & 13 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + (-3)\cdot(-1) & 1\cdot 7 + (-3)\cdot 5 \\ 4\cdot 2 + 2\cdot(-1) & 4\cdot 7 + 2\cdot 5 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 6 & 38 \end{bmatrix}$$

Question: Suppose:

$$f(n) = \begin{cases} f(n-2) & \text{if } n \text{ is even} \\ f(n-2) + 3 & \text{if } n \text{ is odd} \end{cases}$$

Also suppose that $f(0) = 1$ and $f(1) = 4$. Find $f(7)$. [Chapter 4.3 Review]

Answer: Using the recurrence relation, we obtain $f(3) = 7$, $f(5) = 10$, and $f(7) = 13$.

Question: Recursive algorithms [Chapter 4.4 Review]

a. Write a recursive algorithm for finding the sum of the first n even positive integers.

Answer: Let $evensum(n)$ be the sum of the first n even positive integers. A recursive algorithm is:

procedure $evensum\ (n\text{: } integer\ \geq\ 1)$

if $n\ =\ 1$ **then** $evensum\ (n)\ :=\ 2$

else $evensum\ (n)\ :=\ evensum\ (n\ -\ 1)\ +\ 2n$

b. Use mathematical induction to prove that the algorithm in **a** is correct.

Answer: Let $P(n)$ be "$evensum\ (n)$ is the sum of the first n even positive integers."

<u>*Basis Step:*</u> When $n = 1$, the "then" clause of the procedure takes effect, and gives $evensum\ (1)\ =\ 2$, which is the sum of the first even integer.

<u>*Induction Step:*</u> We assume $P(k)$ is true for some $k \geq 1$ and must show that $P(k+1)$ is true. The proposition $P(k)$ states that "$evensum\ (k)$ is the sum of the first k even positive integers". According to the algorithm, because $k + 1 > 1$, the "else" clause is used (with $k\ + 1$ in place of n) to obtain evensum $(k+1)$ and gives

$$\begin{aligned} evensum\ (k\ +\ 1) &= evensum\ (k) +\ 2(k\ +\ 1) \\ &= sum\ of\ the\ first\ k\ even\ integers\ +\ 2(k\ +\ 1) \end{aligned}$$

which is the sum of the first $k\ +\ 1$ even integers. Therefore, the induction step follows. Thus, the Principle of Mathematical Induction proves that the algorithm is correct.

Question: Find the number of integers from 1 to 400 inclusive that are: [Chapter 5.1 Review]

a. divisible by 6 and 8.

Answer: Divisibility by a and b is the same as divisibility by the least common multiple of a and b. Therefore, divisibility by 6 and 8 is the same as divisibility by 24. The answer is $\left\lfloor \frac{400}{24} \right\rfloor = 16$.

b. divisible by 6 or 8.

Answer: We cannot take the number of integers divisible by 6 and add to it the number of integers divisible by 8, because this would count integers such as 24 or 48 twice (because they are divisible by both 6 and 8). We need to use the inclusion-exclusion principle to avoid the "double counting".

Let $A_1 = \{x | 1 \leq x \leq 400, x \text{ is divisible by } 6\}$ and $A_2 = \{x | 1 \leq x \leq 400, x \text{ is divisible by } 8\}$. We want $|A_1 \cup A_2|$. By the inclusion-exclusion principle we have:

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= \left\lfloor \frac{400}{6} \right\rfloor + \left\lfloor \frac{400}{8} \right\rfloor - \left\lfloor \frac{400}{24} \right\rfloor \\ &= 66 + 50 - 16 \\ &= 100 \end{aligned}$$

Question: Each student is classified as a member of one of the following classes: Freshman, Sophomore, Junior, Senior. Find the minimum number of students who must be chosen in order to guarantee that at least eight belong to the same class. [Chapter 5.2 Review]

Answer: The four classes are the pigeonholes. A group of 28 students could have 7 belonging to each class. But if there are 29 students, at least 8 must be members of the same class. Therefore, the minimum number of students who must be chosen is 29.

In other words, we are looking for the minimum number N such that $\left\lceil \frac{N}{4} \right\rceil = 8$. The minimum number is 29.

Question: Find the number of ways to take an ordinary deck of 52 playing cards and break it into: [Chapter 5.3 Review]

a. four equal piles, labeled $A, B, C, and\ D$.

Answer: Each pile must have $52/4 = 13$ cards in it. In sequence, we form pile A, then pile B, then pile C, and finally pile D. There are $C(52,13)$ ways to obtain pile A, $C(39,13)$ ways to obtain pile B, $C(26,13)$ ways to obtain pile C, and $C(13,13) = 1$ way to obtain pile D. Therefore, by the product rule the answer is:

$$C(52,13)\cdot C(39,13)\cdot C(26,13)\cdot C(13,13)=\frac{52!}{13!\cdot 39!}\cdot\frac{39!}{13!\cdot 26!}\cdot\frac{26!}{13!\cdot 13!}\cdot\frac{13!}{13!\cdot 0!}=\frac{52!}{(13!)^4}$$

b. four equal piles that are not labeled.

Answer: If the four piles are not labeled, there is no distinction to be made among piles A, B, C, D. We can permute these in 4! ways. Hence the answer is the answer to part **a** divided by 4!:

$$\frac{C(52,13)\cdot C(39,13)\cdot C(26,13)\cdot C(13,13)}{4!}=\frac{\frac{52!}{13!\cdot 39!}\cdot\frac{39!}{13!\cdot 26!}\cdot\frac{26!}{13!\cdot 13!}\cdot\frac{13!}{13!\cdot 0!}}{4!}=\frac{52!}{(13!)^4\cdot 4!}$$

Question: Four players are playing bridge. In how many ways can they be dealt hands of cards? (In bridge, a hand of cards consists of 13 out of 52 cards.) [Chapter 5.5 Review]

Answer: This is a problem of placing 52 distinguishable objects (the cards) in four distinguishable piles of size 13 (one pile for each of the four players). This can be done is

$$C(52,13)\cdot C(39,13)\cdot C(26,13)\cdot C(13,13)=\frac{52}{13!\,13!\,13!\,13!}$$

Question: If the permutations of $1, 2, 3, 4, 5$ are put in lexicographic order, in what position is the permutation 41253? [Chapter 5.6 Review]

Answer: There are $4! = 24$ permutations of $1, 2, 3, 4, 5$ that begin with 1; these permutations are in positions 1 through 24. Similarly, the permutations in positions 25 through 48 begin with 2 and the permutations in positions 49 through 72 begin with 3. Thus, the first permutation beginning with 4, 41235, is in position 73. Therefore 41253 is in position 74.

Question: Suppose S = {1, 2, . . . , 20}. You select a subset $T \subseteq S$ of size three. Find the probability that T has at least one even number in it. [Chapter 6.1 Review]

Answer: There are $C(20, 3)$ subsets of size three, and choosing any of them is equally likely. It is easiest to use the rule $p(E) = 1 - p(E)$. Let E be the event "T has at least one even number in it". Therefore E is the event "T has only odd numbers in it". We have

$$p(E)=1-p(\bar{E})=1-\frac{C(10,3)}{C(20,3)}\approx 0.895$$

Question: A fair coin is flipped five times. Find the probability of obtaining exactly four heads. [Chapter 6.2 Review]

Answer: This is an example of a sequence of five independent Bernoulli trials. In this example, a success is getting heads. The probability of success is $1/2$ and the probability of failure (getting tails) is $q = 1 - 1/2 = 1/2$. Therefore the probability of getting exactly four heads is

$$b\left(4;\ 5, \frac{1}{2}\right) = C(5,4)\left(\frac{1}{2}\right)^4\left(1 - \frac{1}{2}\right)^1 \approx 0.156$$

Question: It is estimated that a certain disease occurs in 0.1% of the U.S. population. A test that attempts to detect the disease has been developed with the following results: 99.7% of people with the disease test positive for the disease and 0.2% of people without the disease test positive for the disease. (A result that says that a person has the disease when in reality the person does not have the disease is called a "false positive".) Find the probability that a person actually has the disease, given that the person tests positive for the disease. [Chapter 6.3 Review]

Answer: Let A be the event that the person tests positive for the disease and B be the event that the person actually has the disease. We want $p(B|A)$. According to Bayes' Theorem, we have the following. That is, only about one third of the people who test positive for the disease actually have the disease and about two thirds of the people who test positive for the disease are really disease-free.

$$\begin{aligned} p(B\,|A) &= \frac{p(B)p(B|A)}{p(B)p(B|A) + p(\bar{B})p(\bar{B}|A)} \\ &= \frac{\frac{1}{1000}\cdot\frac{997}{1000}}{\frac{1}{1000}\cdot\frac{997}{1000} + \frac{999}{1000}\cdot\frac{2}{1000}} \\ &= \frac{997}{997 + 1998} \\ &\approx 0.333 \\ &= 33.3\% \end{aligned}$$

Question: A 6-sided die has its sides labeled 1, 1, 2, 2, 2, 3. If you roll the die once, what is the expected value of the number that shows? [Chapter 6.4 Review]

Answer: Here $S = \{1, 2, 3\}$ and $p(1) = 2/6$, $p(2) = 3/6$ and $p(3) = 1/6$. For $s \in S$, we let $X(s)$ = the number rolled. Therefore

$$E(x) = \sum_{s\in S} p(s)X(s) = \frac{2}{6}\cdot 1 + \frac{3}{6}\cdot 2 + \frac{1}{6}\cdot 3 = \frac{11}{6}$$