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ICS 241

Recitation Lecture Note #2

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Question: Solve: $a_n = 2a_{n-1} + 3a_{n-2},\ a_0 = 0,\ a_1 = 1$. [Chapter 7.2 Review]

Answer: Using $a_n = r^n$, the following characteristic equation is obtained:

$$r^2 - 2r - 3 = 0$$

The left side factors as $(r-3)(r+1)$, yielding the roots 3 and -1. Hence, the general solution to the given recurrence relation is

$$a_n = c3^n + d(-1)^n.$$

Using the initial conditions $a_0 = 0$ and $a_1 = 1$ yields the system of equations

$$c + d = 0$$

$$3c - d = 1$$

with solution $c = 1/4$ and $d = -1/4$. Therefore, the solution to the given recurrence relation is:

$$a_n = \frac{1}{4} \cdot 3^n - \frac{1}{4} \cdot (-1)^n .$$

Question: Solve: $a_n = -7a_{n-1} - 10a_{n-2},\ a_0 = 3,\ a_1 = 3$. [Chapter 7.2 Review]

Answer: Using $a_n = r^n$, the following characteristic equation is obtained:

$$r^2 + 7r + 10 = 0$$

The left side factors as $(r+5)(r+2)$, yielding the roots -5 and -2. Hence, the general solution to the given recurrence relation is

$$a_n = c(-5)^n + d(-2)^n.$$

Using the initial conditions $a_0 = 3$ and $a_1 = 3$ yields the system of equations

$$c + d = 3$$

$$-5c - 2d = 3$$

with solution $c = -3$ and $d = 6$. Therefore, the solution to the given recurrence relation is:

$$a_n = -3 \cdot (-5)^n + 6 \cdot (-2)^n .$$

Question: Solve: $a_n = 10a_{n-1} - 25a_{n-2},\ a_0 = 3,\ a_1 = 4$. [Chapter 7.2 Review]

Answer: Using $a_n = r^n$, the following characteristic equation is obtained:

$$r^2 - 10r + 25 = 0$$

The left side factors as $(r-5)(r-5)$, yielding the repeated solution 5. Hence, the general solution to the given recurrence relation is

$$a_n = c \cdot 5^n + d \cdot n \cdot 5^n .$$

Using the initial conditions $a_0 = 3$ and $a_1 = 4$ yields the system of equations

$$c + d = 3$$

$$5c + 5d = 4$$

with solution $c = 3$ and $d = -11/5$. Therefore, the solution to the given recurrence relation is:

$$a_n = 3 \cdot 5^n - \frac{11}{5} \cdot n \cdot 5^n .$$

Question: Suppose that the characteristic equation of a linear homogeneous recurrence relation with constant coefficients is: [Chapter 7.2 Review]

$$(r-3)^4(r-2)^3(r+6) = 0$$

Write the general solution of the recurrence relation.

Answer: We can easily obtain the roots for this characteristic equation. They are $3, 2,$ and -6. The multiplicities are 4, 3, and 1 respectly. Therefore according to Theorem 4, the general relation is:

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2 + \alpha_{1,3}n^3\right)3^n + \left(\alpha_{2,0} + \alpha_{2,1}n + \alpha_{2,2}n^2\right)2^n + \alpha_{3,0}(-6)^n$$

Question: Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$ with initial condition $a_0 = 2$. [Chapter 7.2 Review]

Answer: The characteristic equation for the associated homogeneous recurrence relation is $r - 3 = 0$, which has solution $r = 3$. Therefore the general solution to the associated homogeneous recurrence relation is:

$$a_n = \alpha \cdot 3^n .$$

To obtain a particular solution to the given recurrence relation, try $a_n^{(p)} = c \cdot 2^n$, obtaining $c \cdot 2^n = 3c \cdot 2^{n-1} + 2^n$, which yields $c = -2$. Therefore a particular solution is:

$$a_n^{(p)} = -2^{n+1}.$$

Hence, the general solution to the given recurrence relation is:

$$a_n = \alpha \cdot 3^n - 2^{n+1}.$$

The initial condition $a_0 = 2$ gives $2 = \alpha \cdot 1 - 2$, or $\alpha = 4$. Therefore the solution to the given nonhomogeneous recurrence relation is:

$$a_n = 4 \cdot 3^n - 2^{n+1}.$$

Question: Solve the recurrence relation $a_n = 8a_{n-1} - 12a_{n-2} + 3n$ with initial condition $a_0 = 1$ and $a_1 = 5$. [Chapter 7.2 Review]

Answer: The characteristic equation for the associated homogeneous recurrence relation is $r^2 - 8r + 12 = 0$, which can be written as $(r-2)(r-6) = 0$. So it has solutions $r_1 = 2$ and $r_2 = 6$. Therefore the general solution to the associated homogeneous recurrence relation is:

$$a_n = a \cdot 2^n + b \cdot 6^n.$$

To obtain a particular solution to the given recurrence relation, try $a_n^{(p)} = c \cdot n + d$, obtaining:

$$c \cdot n + d = 8[c(n-1) + d] - 12[c(n-2) + d] + 3n$$

The coefficient of n-term and the constant term must each equal 0. Therefore, we have

$$c - 8c + 12c - 3 = 0$$

$$d + 8c - 8d - 24c + 12d = 0$$

Hence, we have $c = 3/5$ and $d = 48/25$. Therefore:

$$a_n = a \cdot 2^n + b \cdot 6^n + \frac{3}{5}n + \frac{48}{25}.$$

The initial condition $a_0 = 1$ and $a_1 = 5$, yields the system of equations:

$$a6^0 + b2^0 + \frac{3}{5} \cdot 0 + \frac{48}{25} = 1$$

$$a6^1 + b2^1 + \frac{3}{5} \cdot 1 + \frac{48}{25} = 5$$

and the solution is found to be $a = 27/25$ and $b = -2$. Therefore the solution to the given recurrence relation is:

$$\frac{27}{25}6^n - 2^{n+1} + \frac{3}{5} \cdot n + \frac{48}{25}.$$

Question: Suppose $f(n) = 3f(n/2) + 4$ and $f(1) = 5$ Find $f(8)$. [Chapter 7.3 Review]

Answer: $f(2) = 3f(2/2) + 4 = 3 \cdot 5 + 4 = 19,$

$f(4) = 3f(4/2) + 4 = 3 \cdot 19 + 4 = 61,$

$f(8) = 3f(8/2) + 4 = 3 \cdot 61 + 4 = 187\,.$

Question: Suppose $f(n) = 5f(n/2) + 2n - 1$ and $f(4) = 40$ Find $f(1)$. [Chapter 7.3 Review]

Answer: $f(4) = 5f(4/2) + 2 \cdot 4 - 1 = 40 \quad \Longrightarrow \quad f(2) = 33/5$

$f(2) = 5f(2/2) + 2 \cdot 2 - 1 = 33/5 \quad \Longrightarrow \quad f(1) = 18/25\,.$

Question: Suppose $f(n) = 2f(n/3) + 3$ Find a big-Oh function for f. [Chapter 7.3 Review]

Answer: According to Theorem 1 of Section 7.3, $f(n)$ is $O\left(n^{\log_3 2}\right)$.

Question: Prove by induction the standard formula for the sum of the first n terms of an arithmetic series: [Prove by Induction Problem. Homework #0 Review]

$$\sum_{k=1}^{n} [a + (k-1)d] = an + \frac{1}{2}n(n-1)d$$

Answer: Let $P(n)$ be the proposition: $\sum_{k=1}^{n-1}[a + (k-1)d] = a(n-1) + \frac{1}{2}(n-1)(n-2)d$

Basis Step: When $n = 1$, the left side is $\sum_{k=1}^{n}[a + (k-1)d] = a$; the right side is $an + \frac{1}{2}n(n-1)d = a$. Therefore the base case has been proved.

Inductive Step: $P(n-1) \to P(n)$Assume the formula equation holds for $n-1$, which indicates $\sum_{k=1}^{n-1}[a + (k-1)d] = a(n-1) + \frac{1}{2}(n-1)(n-2)d.$

$$\begin{aligned}
\sum_{k=1}^{n} [a + (k-1)d] &= \sum_{k=1}^{n-1} [a + (k-1)d] + [a + (n-1)d] \\
&= a(n-1) + \frac{1}{2}(n-1)(n-2)d + [a + (n-1)d] \\
&= an + \frac{1}{2}(n^2 - 3n + 2)d + (n-1)d \\
&= an + \frac{1}{2}(n^2 - n + 1)d \\
&= an + \frac{1}{2}n(n-1)d\,.
\end{aligned}$$

Therefore: We have proved if the current step holds, the next step holds as well. Therefore we have proved the basis and the induction step. The input formula for the sum of the first n terms of an arithmetic series $\sum_{k=1}^{n}[a+(k-1)d] = an + \frac{1}{2}n(n-1)d$ holds.

Question: For the sequence of Fibonacci numbers $f_0, f_1, f_2, \dots (0, 1, 1, 2, 3, 5, 8, 13, \dots)$, prove that:

[Another Prove by Induction Problem]

$$f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$$

Answer: Let $P(n)$ be: $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$

Basis Step: $P(0)$ states that $f_0 = f_1 - 1$, which is true because $f_0 = 0$ and $f_1 - 1 = 1 - 1 = 0$.

Induction Step: $P(k) \to P(k+1)$: Suppose that $P(k)$ is true; i.e., $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2k} = f_{2k+1} - 1$. We must show that $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2(k+1)} = f_{2(k+1)+1} - 1$, i.e., $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2k+2} = f_{2k+3} - 1$:

$$\begin{aligned} f_0 + f_2 + f_4 + f_6 + \cdots + f_{2k+2} &= (f_0 + f_2 + f_4 + f_6 + \cdots + f_{2k}) + f_{2k+2} \\ &= (f_{2k+1} - 1) + f_{2k+2} \\ &= f_{2k+1} + f_{2k+2} - 1 \\ &= f_{2k+3} - 1\,. \end{aligned}$$

Therefore: We have proved if the current step holds, the next step holds as well. Therefore we have proved the basis and the induction step. For the sequence of Fibonacci numbers $f_0, f_1, f_2, \dots (0, 1, 1, 2, 3, 5, 8, 13, \dots)$, $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$ holds.