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ICS 241

Recitation Lecture Note #3

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Question: Use Master Theorem to determine the runtime of the following functions. [Chapter 7.3 Review]

Review: Here is a little review of the Master's Theorem. Let $f(n)$ be a monotonically increasing function that satisfies $T(n) = aT(n/b) + f(n)$ and $T(1) = c$. Where $a \geq 1,\ b > 1,\ c > 0$. If $f(n)$ is a $\Theta(n^d)$ function, where $a \geq 0$ then:

$$T(n) = \begin{cases} \Theta(n^d) & if\ a < b^d \\ \Theta(n^d logn) & if\ a = b^d \\ \Theta\left(n^{\log_b a}\right) & if\ a > b^d \end{cases}$$

There are several prerequisite to keep in mind.

1. $T(n)$ has to be monotone, e.g. $T(n) = \sin(x)$ is not suitable to use this theorem.
2. $f(n)$ has to be a polynomial function, e.g. $T(n) = 2T(n/2) + 2^n$.
3. b need to be able to expressed as a constant, e.g. $T(n) = T(\sqrt{n})$.

a. Let $\boldsymbol{T(n) = 2T\left(\frac{n}{2}\right) + \frac{1}{2}n^2 + n}$. What are the parameters?

Answer: $a = 1; b = 2; d = 2$. Since $1 < 2^2$, Case 1 applies. Thus we conclude that $T(n) \in \Theta(n^d) = \Theta(n^2)$.

b. Let $\boldsymbol{T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42}$. What are the parameters?

Answer: $a = 2; b = 4; d = 1/2$. Since $2 < 4^{1/2}$, Case 2 applies. Thus we conclude that $T(n) \in \Theta(n^d logn) = \Theta(\sqrt{n} \cdot logn)$.

c. Let $\boldsymbol{T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1}$. What are the parameters?

Answer: $a = 3; b = 2; d = 1$. Since $3 < 2^1$, Case 3 applies. Thus we conclude that $T(n) \in \Theta\left(n^{\log_b a}\right) = \Theta(n^{\log_2 3})$.

Question: How many positive integers less than or equal to 100 are divisible by 6 or 9? [Chapter 7.5 Review]

Answer: Let A be the set of integers from 1 to 100 divisible by 6 and let B be the set of integers from 1 to 100 divisible by 9. By the inclusion-exclusion principle, the number of positive integers from 1 to 100 divisible by 6 or 9 is

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= \left\lfloor \frac{100}{6} \right\rfloor + \left\lfloor \frac{100}{9} \right\rfloor - \left\lfloor \frac{100}{18} \right\rfloor \\ &= 16 + 11 - 5 \\ &= 22\,.\end{aligned}$$

Question: How many positive integers less than or equal to 100 are relatively prime to 15? [Chapter 7.5 Review]

Answer: An integer is relatively prime to 15 if and only if it is not divisible by 3 and not divisible by 5. Let A be the set of integers from 1 to 100 divisible by 3 and let B be the set of integers from 1 to 100 divisible by 5. Then the number of integers that are not relatively prime to 15 is

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor \\ &= 33 + 20 - 6 \\ &= 47\,.\end{aligned}$$

Hence, the number of integers from 1 to 100 that are relatively prime to 15 is

$$|\overline{A \cup B}| = 100 - 47 = 53\,.$$

Question: Find the number of elements in $A_1 \cup A_2 \cup A_3 \cup A_4$ if each set has size 50, each intersection of two sets has size 30, each intersection of three sets has size 10, and the intersection of all four sets has size 2. [Chapter 7.5 Review]

Answer: Using the inclusion-exclusion principle,

$$|A_1 \cup A_2 \cup A_3 \cup A_4|$$

$$= \sum_{i=1}^{4} |A| - \sum_{i \neq j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= 4 \cdot 5 - \binom{4}{2} \cdot 30 + \binom{4}{3} \cdot 10 - 2$$

$$= 200 - 1800 + 40 - 2$$

$$= 58\,.$$

Question: Inclusion and Exclusion problems. [Chapter 7.5 Review]

a. Find the number of permutations of $\mathbf{1, 2, \cdots, 8}$ that begin with $\mathbf{52}$ or end with $\mathbf{387}$.

Answer: Let A be the set of permutations of $1, 2, \cdots, 8$ that begin with 52 and let B be the set of permutations of $1, 2, \cdots, 8$ that end with 387. Using the inclusion-exclusion principle,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 6! + 5! - 3! \\ &= 720 + 120 - 6 \\ &= 834\,. \end{aligned}$$

b. Find the number of permutations of $\mathbf{1, 2, \cdots, 8}$ that begin with $\mathbf{52}$ or end with $\mathbf{327}$. [Chapter 7.5 Review]

Answer: Let A be the set of permutations of $1, 2, \cdots, 8$ that begin with 52 and let B be the set of permutations of $1, 2, \cdots, 8$ that end with 327. In this case $A \cap B = \emptyset$ because the digit 2 cannot occur in both the string 52 and 327. Therefore,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 6! + 5! \\ &= 720 + 120 \\ &= 840\,. \end{aligned}$$

Question: Find the number of permutations of all 26 letters of the alphabet that contain at least one of the words CAR, CARE, SCARE, SCARED. [Chapter 7.5 Review]

Answer: Let $A, B, C,$ and D be the sets of permutations of the 26 letters of the alphabet that contain the words CAR, CARE, SCARE, and SCARED, respectively. Then $D \subseteq C \subseteq B \subseteq A$. Hence,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A| = 24!$$

Question: Suppose $|U| = n$ and A and B are subsets of U such that $|A| > n/2$, and $B > n/2$. Prove that $A \cap B \neq \emptyset$. [Chapter 7.5 Review]

Answer: Suppose $A \cap B = \emptyset$. By the inclusion-exclusion principle

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= |A| + |B| \\ &\geq \frac{n}{2} + \frac{n}{2} \\ &= n \end{aligned}$$

Therefore $|A \cup B| > n$. But this is not possible because $A \cup B \subseteq U$ and every subset of U has size at most n. Therefore $A \cap B \neq \emptyset$.

Question: Find a recurrence relation for the number of ways to climb n stairs [Homework 1 Review]

a. if the person climbing the stairs can take one, two, or three stairs at a time.

Answer: Let a_n be the number of ways to climb n stairs. To finish climbing n stairs, there are three ways to divide this problem. A person can start with a step of one stair and then climb $n - 1$ stairs, and this can be done in a_{n-1} ways. This person can also start with a step of two stairs and then climb $n - 2$ stairs, and this can be done in a_{n-2} ways. This person can also start with a step of three stairs and then climb $n - 3$ stairs, and this can be done in a_{n-3} ways.

From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

b. What are the initial conditions?

Answer: The initial conditions are $a_0 = 1$, $a_0 = 1$, $a_2 = 2$.

There is one way to climb no stairs, do nothing; There is one way to climb one stair, a step of one stair; There is two ways to climb two stairs, either two steps of one stair or one step of two stairs.

Double Check: $a_3 = a_0 + a_1 + a_2 = 4$. There are four ways to climb three stairs. One can climb three steps of one stair. One can climb a step of one stair followed by a step of two stairs. One can climb a step of two stairs followed by a step of one stair. Once can also take one step of three stairs. Therefore $a_3 = 4$ is indeed the solution for $n = 3$.

c. How many ways can this person climb a flight of eight stairs.

Answer: $a_0 = 1$

$a_1 = 1$

$a_2 = 2$

$a_3 = a_0 + a_1 + a_2 = 4$

$a_4 = a_1 + a_2 + a_3 = 7$

$a_5 = a_2 + a_3 + a_4 = 13$

$a_6 = a_3 + a_4 + a_5 = 24$

$a_7 = a_4 + a_5 + a_6 = 44$

$a_8 = a_5 + a_6 + a_7 = 81$.