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ICS 241

Recitation Lecture Note #4

Exam #1 Review

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Exam #1 Review

Question: Verify that the program segment

if $x < y$ **then**

$min := x$

else

$min := y$

is correct with respect to the initial assertion T and the final assertion $(x \leq y \wedge min = x) \vee (x > y \wedge min = y)$. [Chapter 4.5 Review]

Answer: There are three cases. If $x < y$ initially, then min is set equal to x, so $(x \leq y \wedge min = x)$ is true. If $x = y$ initially, then min is set equal to y (while equals x), so again $(x \leq y \wedge min = x)$ is true. Finally, if $x > y$ initially, then min is set equal to y, so $(x > y \wedge min = y)$ is true. Hence in all cases the disjunction $(x \leq y \wedge min = x) \vee (x > y \wedge min = y)$ is true.

Question: Suppose $b_n = b_{n-1} + n - 2^n$ and $b_0 = 5$. [Chapter 7.1 Review]

a. Find b_{n-1} in terms of b_{n-2}.

Answer: The recurrence relation for b_n the previous term (which is b_{n-1}), adds the subscript number of b_n (which is n), and subtracts 2 raised to the power of the subscript of b_n (which is 2^n).

The term b_{n-1} is obtained in the same way: the previous term (which is b_{n-2}), add the subscript number of b_{n-1} (which is $n - 1$), and subtract 2 raised to the power of the subscript of b_{n-1} (which is 2^{n-1}). Therefore $b_{n-1} = b_{n-2} + (n - 1) - 2^{n-1}$.

b. Find b_n in terms of b_{n-2}.

Answer: To obtain b_n in terms of b_{n-2}, we first use the recurrence relation to obtain b_n in terms of b_{n-1} and then use part **a**. to obtain b_{n-1} in terms of b_{n-2}:

$$
\begin{aligned}
b_n &= b_{n-1} + n - 2^n \\
&= [b_{n-2} + (n-1) - 2^{n-1}] + n - 2^n \\
&= b_{n-2} + (n-1) + n - 2^{n-1} - 2^n \,.
\end{aligned}
$$

c. Find b_n in terms of b_{n-3}.

Answer: To obtain b_n in terms of b_{n-3}, we can use the recurrence equation for b_{n-2} and part **b.** The recurrence relation for b_{n-2} is $b_{n-2} = b_{n-3} + (n-2) - 2^{n-2}$. Substituting for b_{n-2} into the result in part **b.**, we have

$$
\begin{aligned}
b_n &= b_{n-2} + (n-1) + n - 2^{n-1} - 2^n \\
&= [b_{n-3} + (n-2) - 2^{n-2}] + (n-1) + n - 2^{n-1} - 2^n \\
&= b_{n-3} + (n-2) + (n-1) + n - 2^{n-2} - 2^{n-1} - 2^n \,.
\end{aligned}
$$

d. Use parts **b.** and **c.** to conjecture a formula for b_n.

Answer: Mentally continuing the pattern in part **c.**, it seems reasonable to guess that the first term becomes b_0, the sum $(n-2) + (n-1) + n$ becomes $1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$, and the sum of the powers of 2 being subtracted becomes $-2^1 - 2^2 - 2^3 - \cdots - 2^{n-2} - 2^{n-1} - 2^n$. Thus, it is reasonable to conjecture that

$$
\begin{aligned}
b_n &= b_0 + [1 + 2 + 3 + \cdots + (n-1) + n] - [2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} + 2^n] \\
&= 5 + \frac{n(n+1)}{2} - 2^{n+1} + 2 \,.
\end{aligned}
$$

where summation formulas were used at the last step to replace $1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$ and $2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} + 2^n$.

<u>*Double check:*</u> Apply both the recurrence relation and the formula to compute the first three terms of this series:

$b_1 = b_0 + 1 - 2^1 = 4$	$b_1 = 5 + 1 - 2^2 + 2 = 4$	***check!***
$b_2 = b_1 + 2 - 2^2 = 2$	$b_2 = 5 + 3 - 2^3 + 2 = 2$	***check!***
$b_3 = b_2 + 3 - 2^3 = -3$	$b_3 = 5 + 6 - 2^4 + 2 = -3$	***check!***

Therefore we've came up with our conclusion that the solutions for this recurrence relation is

$$
b_n = 5 + \frac{n(n+1)}{2} - 2^{n+1} + 2 \,.
$$

Question: Solve: $a_n = 3a_{n-1} + 1$, $a_0 = 4$, by substituting for a_{n-1}, then a_{n-2}, etc. [Chapter 7.1 Review]

Answer: Beginning with $a_n = 3a_{n-1} + 1$ and substituting for a_{n-1}, then a_{n-2}, then a_{n-3} etc., yields:

$$\begin{aligned} a_n &= 3a_{n-1} + 1 \\ &= 3(3a_{n-2} + 1) + 1 = 3^2 a_{n-2} + 3 + 1 \\ &= 3^2(3a_{n-3} + 1) + 3 + 1 = 3^3 a_{n-3} + 3^2 + 3 + 1 \\ &= 3^3(3a_{n-4} + 1) + 3^2 + 3 + 1 = 3^4 a_{n-4} + 3^3 + 3^2 + 3 + 1 \\ &\vdots \\ &= 3^n a_{n-n} + (3^n + 3^{n-1} + \cdots + 3 + 1) \\ &= 3^n a_0 + (3^n + 3^{n-1} + \cdots + 3 + 1) \\ &= 4 \cdot 3^n + \frac{3^n - 1}{2} \\ &= \frac{9}{2} \cdot 3^n - \frac{1}{2}. \end{aligned}$$

Question: Solve: $a_n = 3a_{n-1} + 4a_{n-2}$, $a_0 = a_1 = 5$. [Chapter 7.2 Review]

Answer: Using $a_n = r^n$, the following characteristic equation is obtained:

$$r^2 - 3r - 4 = 0$$

The left side factors as $(r-4)(r+1)$, yielding the roots 4 and -1. Hence, the general solution to the given recurrence relation is

$$a_n = c4^n + d(-1)^n.$$

Using the initial conditions $a_0 = 0$ and $a_1 = 1$ yields the system of equations

$$c + d = 5$$

$$4c - d = 5$$

with solution $c = 2$ and $d = 3$. Therefore, the solution to the given recurrence relation is:

$$a_n = 2 \cdot 3^n + 3 \cdot (-1)^n.$$

Question: Solve the recurrence relation $a_n = 2a_{n-1} + 4a_{n-2} - 8a_{n-3}$. Write the general solution of the recurrence relation. [Chapter 7.2 Review]

Answer: The characteristic equation for this homogeneous recurrence relation is $r^3 - 2r^2 - 4r + 8 = 0$,

$$\begin{aligned} r^3 - 2r^2 - 4r + 8 &= r^3 - 4r - (2r^2 - 8) \\ &= r(r^2 - 4) - 2(r^2 - 4) \\ &= (r^2 - 4)(r - 2) \\ &= (r + 2)(r - 2)(r - 2) \\ &= (r - 2)^2(r + 2)\,. \end{aligned}$$

The characteristic equation can be written as $(r - 2)^2(r + 2) = 0$. So it has solutions $r_1 = 2$ and $r_2 = -2$. The multiplicities of these two roots are 2 and 1 respectively. Therefore the general solution to the associated homogeneous recurrence relation is:

$$a_n = (a + b \cdot n) \cdot 2^n + c \cdot (-2)^n\,.$$

Question: Solve the recurrence relation $a_n = 2a_{n-1} + 2n^2$ with initial condition $a_1 = 4$. [Chapter 7.2 Review]

Answer: The characteristic equation for the associated homogeneous recurrence relation is $r - 2 = 0$. So it has solutions $r = 2$ Therefore the general solution to the associated homogeneous recurrence relation is:

$$a_n = a \cdot 2^n\,.$$

To obtain a particular solution to the given recurrence relation, try $a_n^{(p)} = b \cdot n^2 + c \cdot n + d$, obtaining:

$$\begin{aligned} b \cdot n^2 + c \cdot n + d &= 2b \cdot (n-1)^2 + 2c \cdot (n-1) + 2d + 2n^2 \\ &\Longrightarrow (b+2)n^2 + (c - 4b)n + 2b - 2c + d = 0 \end{aligned}$$

The coefficient of n^2, n terms and the constant term must each equal 0. Therefore, we have

$$b + 2 = 0$$

$$c - 4b = 0$$

$$2b - 2c + d = 0$$

Hence, we have $b = -2$ $c = -8$ and $d = -12$. Therefore:

$$a_n = a \cdot 2^n - 2n^2 - 8n - 12.$$

The initial condition $a_1 = 4$, yields the system of equations coefficient:

$$a_1 = a \cdot 2^1 - 2 \cdot 1^2 - 8 \cdot 1 - 12 = 4$$

$$\Longrightarrow a = 13\,.$$

The coefficient is found to be $a = 13$. Therefore the solution to the given recurrence relation is:

$$a_n = 13 \cdot 2^n - 2n^2 - 8n - 12.$$

Question: A recursive algorithm for finding the maximum of a list of numbers divides the list into three equal (or nearly equal) parts, recursively finds the maximum of each sub-list, and then finds the largest of these three maxima. Let $f(n)$ be the total number of comparisons needed to find the maximum of a list of n numbers (n a power of 3). Set up a recurrence relation for $f(n)$ and give a big-oh estimate for f. [Chapter 7.3 Review]

Answer: A recurrence relation for the number of steps in this algorithm with an input of size $n > 1$ (n a power of 3) is:

$$f(n) = 3f\left(\frac{n}{3}\right) + 2\,.$$

(Assuming that two operations are required to compare the three maxima). Using Theorem 1 of Section 7.3, $f(n)$ is$O\left(n^{\log_3 3}\right)$. But $n^{\log_3 3} = n^1 = n$. Therefore $f(n)$ is $O(n)$.

We can also apply Mater Theorem, we have $a = 3; b = 3; d = 0$. Since $3 > 3^0$, Case 1 applies. Thus we conclude that $f(n) \in O\left(n^{\log_3 3}\right) = O(n^1) = O(n)$.

Of course, this solution makes sense. If we were to pick out the largest number from a list of random numbers, we would go though the list one by one, and remember the largest number we've seen. If the current number is larger than what is in our mind, we would instead remember this new maximum. By the time we reach the end of the list, we would have obtained the maximum number of this list. Therefore, looping through the list once is enough, which is the order of n.

Question: Find the number of positive integers not exceeding 1000 that tar either the square or the cube of an integer. [Chapter 7.5 Review]

Answer: Let A be the set of integers from 1 to 1000 that is a square of an integer. Let B be the set of integers from 1 to 1000 that is cube of an integer.

The size of set A is $\left\lfloor \sqrt[2]{1000} \right\rfloor = 31$. The size of set B is $\left\lfloor \sqrt[3]{1000} \right\rfloor = 10$. Furthermore there are $\sqrt[6]{1000} = 3$ numbers that are both squares and cubes, such as 2^6. Therefore the solution is:

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \left\lfloor \sqrt[2]{1000} \right\rfloor + \left\lfloor \sqrt[3]{1000} \right\rfloor - \sqrt[6]{1000} \\ &= 31 + 10 - 3 = 38 \end{aligned}$$

Question: Find the number of solutions of the equation $x_1 + x_2 + x_3 = 13$, where $x_i, i = 1, 2, 3$, are nonnegative integers less than 6. [Chapter 7.6 Review]

Answer: To apply the principle of inclusion-exclusion, let a solution have property P_1 if $x_1 \geq 6$, property P_2 if $x_2 \geq 6$, and property P_3 if $x_3 \geq 6$. The number of solutions satisfying the inequalities $x_1 < 6, x_2 < 6$, and $x_3 < 6$ is

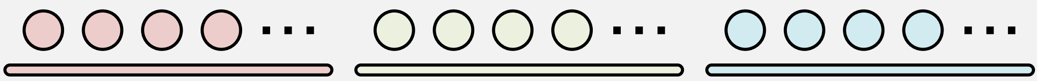
$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_3) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

N = total number of solutions = $C(3 + 13 - 1, 13) = 105$
$N(P_1)$ = number of solutions with $x_1 \geq 6 = C(3 + 7 - 1, 7) = 36$
$N(P_2)$ = total number of solutions $x_2 \geq 6 = C(3 + 7 - 1, 7) = 36$
$N(P_3)$ = total number of solutions $x_3 \geq 6 = C(3 + 7 - 1, 7) = 36$
$N(P_1P_2)$ = total number of solutions $x_1 \geq 6$ and $x_2 \geq 6 = C(3 + 1 - 1, 1) = 3$
$N(P_1P_3)$ = total number of solutions $x_1 \geq 6$ and $x_3 \geq 6 = C(3 + 1 - 1, 1) = 3$
$N(P_2P_3)$ = total number of solutions $x_2 \geq 6$ and $x_3 \geq 6 = C(3 + 1 - 1, 1) = 3$
$N(P_1P_2P_3)$ = total number of solutions $x_1 \geq 6, x_2 \geq 6, x_3 \geq 6 = 0$

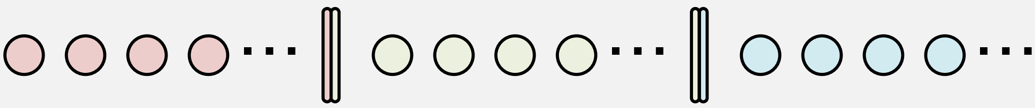
Inserting these quantities into the formula for $N(P_1'P_2'P_3')$ shows that the number of solutions with that $x_1 \leq 6, x_2 \leq 6, x_3 \leq 5$, and $x_4 \leq 8$ equals: $N(P_1'P_2'P_3') = 105 - 36 - 36 - 36 + 3 + 3 + 3 - 0 = 6$

Reasoning: Let's look at this problem: We have three distinctive boxes. Each of them contains endless of balls. How many ways are there to pick out 13 balls from these three boxes? Does it look familiar? Yes, this is equivalent to the total solution, $N = C(3 + 13 - 1, 13) = 105$.

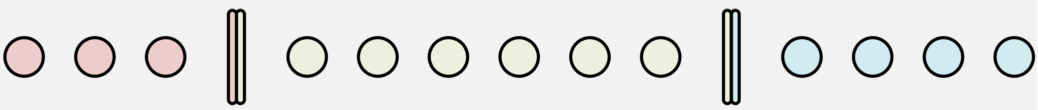
But where does the formula come from? Let's illustrate this model. We have three distinguishable boxes and each containing unlimited balls:



We can simplify it a little bit, we have:

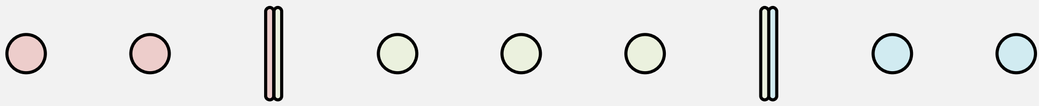


For example, one of our solutions can be expressed like:



As we can see, it has turned into a problem of arranging 15 objects. Two of them are the box separators, and thirteen of them are the balls. The layout of the separators determines the colors of the balls. How many balls come from the red group, how many of them come from the green group, and so on are completely depended on where we put the separators. Therefore, the problem is now: how many ways of choosing 2 object from 15 object, which is by definition $C(15, 2) = C(3 + 15 - 1, 2)$. It can also be expressed as $C(15, 13) = C(3 + 13 - 1, 13)$.

Then, we have some extra conditions to satisfy. "x_1 is a nonnegative integer less than 6" is equivalent to "The red box contains only 5 balls". In order to compute this, we can compute the complementary set, which is "The red contains at least 6 balls". If the red box contains at least 6 balls, there are $13 - 6 = 7$ balls left to distribute:



According to our previous reasoning, there are $C(3 + 7 - 1, 2) = C(9,2)$, or $C(3 + 7 - 1, 7) = C(9,7)$ ways to arrange 7 balls and 2 separators.

After computing overall complementary set of "$x_i, i = 1, 2, 3$, are nonnegative integers less than 6", which is done applying Inclusion-Exclusion formula, we obtain the solution by subtracting it from the total solution $C(15, 13)$.

Exam #1 Review

Question: Let R be the following relation defined on the set $\{a, b, c, d\}$:

$$R = \{(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, b), (c, c), (d, b), (d, d)\}.$$

Determine whether R is [Chapter 8.1 Review]

a. Reflexive

Answer: R is reflexive because R contains (a, a), (b, b), (c, c), and (d, d).

b. symmetric

Answer: R is not symmetric because $(a, c) \in R$, but $(c, a) \notin R$.

c. antisymmetric

Answer: R is not antisymmetric because $(b,c) \in R$, and $(c,b) \in R$,.but $b \neq c$.

d. transitive

Answer: R is not transitive because, for example, $(a,c) \in R$ and $(c,b) \in R$, but $(a,b) \notin R$.

Question: Let R be the following relation on the set of real numbers:

$$aRb \leftrightarrow \lfloor a \rfloor = \lfloor b \rfloor, \text{where } \lfloor x \rfloor \text{ is the floor of } x.$$

Determine whether R is [Chapter 8.1 Review]

a. Reflexive

Answer: R is reflexive because $\lfloor a \rfloor = \lfloor a \rfloor$ is true from all real numbers.

b. symmetric

Answer: R is symmetric suppose $\lfloor a \rfloor = \lfloor b \rfloor$; then $\lfloor b \rfloor = \lfloor a \rfloor$.

c. antisymmetric

Answer: R is not antisymmetric: we can have aRb and bRa for distinct a and b. For example, $\lfloor 1.1 \rfloor = \lfloor 1.2 \rfloor$.

d. transitive

Answer: R is transitive, suppose $\lfloor a \rfloor = \lfloor b \rfloor$ and $\lfloor b \rfloor = \lfloor c \rfloor$; from transitivity of equality of real numbers, it follows that $\lfloor a \rfloor = \lfloor c \rfloor$.