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ICS 241

Recitation Lecture Note #6

October 2, 2009

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Question: Let R be the relation on $\{1, 2, 3, 4\}$ such that [Chapter 8.4 Review]

$$R = \{(1,1), (1,4), (2,3), (3,1), (3,3), (4,4)\}$$

a. Find the reflexive closure of R.

Answer: $R = \{(1,1), (1,4), (2,2), (2,3), (3,1), (3,3), (4,4)\}$.

b. Find the symmetric closure of R.

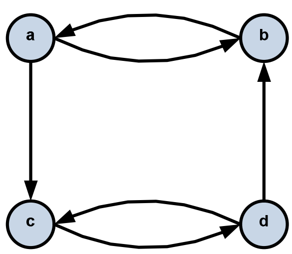
Answer: $R = \{(1,1), (1,3), (1,4), (2,3), (3,1), (3,2), (3,3), (4,1), (4,4)\}$.

c. Find the transitive closure of R.

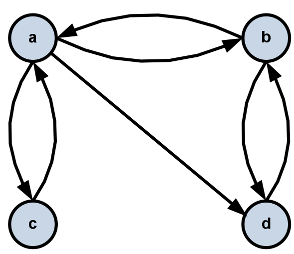
Answer: $R = \{(1,1), (1,4), (2,1), (2,3), (2,4), (3,1), (3,3), (3,4), (4,4)\}$.

Question: Solve the following problems on the following directed graphs. [Chapter 8.4 Review]

A.

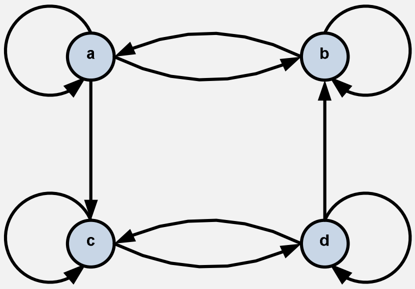


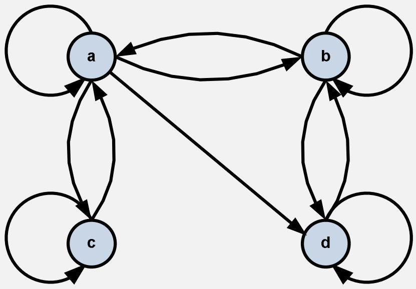
B.



a. Draw the directed graphs of the reflexive closure of the relations with the directed graphs above

Answer: A. B.





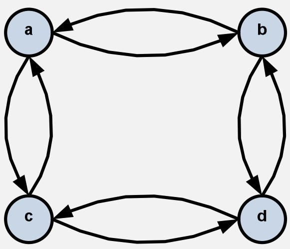
Let's verify the solution by representing the relations using the zero-one matrices.

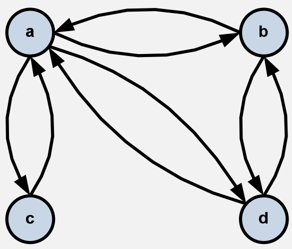
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow A_{relexive\ closure} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow B_{relexive\ closure} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

b. Draw the directed graphs of the symmetric closure of the relations with the directed graphs.

Answer: A. B.





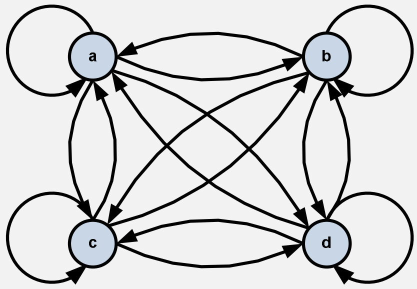
Let's verify the solution by representing the relations using the zero-one matrices.

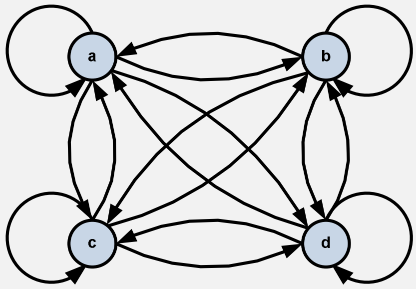
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow A_{symmetric\ closure} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow B_{symmetric\ closure} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

c. Draw the directed graphs of the transitive closure of the relations with the directed graphs.

Answer: A. B.





Let's verify the solution by representing the relations using the zero-one matrices.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} A^3 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} A^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A_{transitive\ closure} = A \cup A^2 \cup A^3 \cup A^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} B^2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} B^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} B^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow B_{transitive\ closure} = B \cup B^2 \cup B^3 \cup B^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Question: Use Algorithm 1 and Warshall's Algorithm to find the transitive closure of the relations on the following relation on $\{1, 2, 3, 4\}$. [Chapter 8.4 Review]

$$\{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$$

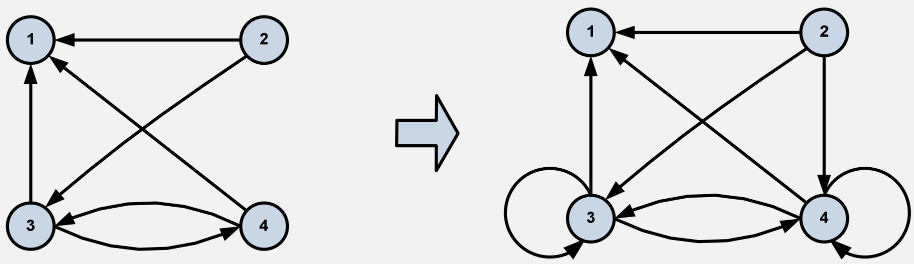
a. Solving the problem using Algorithm 1.

Answer:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A \cup A^2 \cup A^3 \cup A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Let's verify the solution by representing the relation using the directed graph.



b. Solving the problem using Warshall's Algorithm.

Review: Warshall's Algorithm described in pseudocode:

Procedure Warshall (M_R : $n \times n$ zero-one matrix)
$W := M_R$
for $k := 1$ to n
begin
    for $i := 1$ to n
    begin
        for $j := 1$ to n
        $w_{ij} := w_{ij} \vee \left(w_{ik} \wedge w_{kj}\right)$
    end
end $\{W = [w_{ij}] \text{ is } M_{R^*}\}$

Answer:

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{Start with } k = 1, i = 1, j = 1$$

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 1, i = 1$$

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 1, i = 2$$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 1, i = 3$$

$$W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 1, i = 4$$

$$W_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 2, i = 1$$

$$W_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 2, i = 2$$

$$W_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 2, i = 3$$

$$W_8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 2, i = 4$$

$$W_9 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 3, i = 1 \qquad W_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 3, i = 2$$

$$W_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{after } k = 3, i = 3 \qquad W_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{after } k = 3, i = 4$$

$$W_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{after } k = 4, i = 1 \qquad W_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{after } k = 4, i = 2$$

$$W_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{after } k = 4, i = 3 \qquad W_{16} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{after } k = 4, i = 4\,.$$

So the solution for the transitive closure of the relation $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, which is of course the same as the solution applying Algorithm 1.

Question: Find the transitive closure of relation $\{(1,2),(2,3),(3,1)\}$ on $\{a,b,c\}$ using Warshall's Algorithm. [Chapter 8.4 Review]

Answer:

$$W_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{Start with } k = 1, i = 1, j = 1$$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{after } k = 1, i = 1 \qquad W_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{after } k = 1, i = 2$$

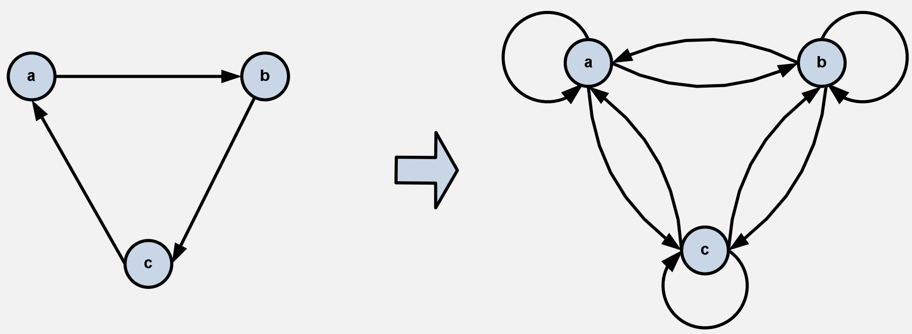
$$W_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{after } k = 1, i = 3 \qquad W_4 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{after } k = 2, i = 1$$

$$W_5 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{after } k = 2, i = 2 \qquad W_6 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{after } k = 2, i = 3$$

$$W_7 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{after } k = 3, i = 1 \qquad W_8 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{after } k = 3, i = 2$$

$$W_9 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{after } k = 3, i = 3\,.$$

So the solution for the transitive closure of the relation $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, which is obvious if we represent the relation with a directed graph.



The connectivity of the original graph is clear. All three vertices are connected with each other by taking different paths. The transitive closure is, therefore, should be a complete directed graph with all possible edges presenting. In other words, the zero-one matrix representing should have 1's in all positions.

Question: We have that the following relation on the set of real numbers: [Chapter 8.5 Review]

$$aRb \leftrightarrow \lfloor a \rfloor = \lfloor b \rfloor, \text{where } \lfloor x \rfloor \text{ is the floor of } x\,.$$

a. Verify that the relation is an equivalence relation.

Answer: R is reflexive because $\lfloor a \rfloor = \lfloor a \rfloor$ is true for all real numbers; R is symmetric because if $\lfloor a \rfloor = \lfloor b \rfloor$, then $\lfloor a \rfloor = \lfloor b \rfloor$ is true for all real numbers; R is transitive. Suppose $\lfloor a \rfloor = \lfloor b \rfloor$ and $\lfloor b \rfloor = \lfloor c \rfloor$, from transitivity of equality of real numbers, it follows that $\lfloor a \rfloor = \lfloor c \rfloor$. Therefore we've proven all three properties of an equivalence relation, $aRb \leftrightarrow \lfloor a \rfloor = \lfloor b \rfloor$ is an equivalence relation.

b. Describe the equivalence classes arising from the equivalence relation.

Answer: Two real numbers, a and b, are related if they have the same floor. This happens if and only if a and b lie in the same interval $[n, n+1)$ where n is an integer. That is the equivalence classes are the intervals $\cdots, [-2,-1), [-1,0), [0,1), [1,2), \cdots$.

Question: Let A be the set of all points in the plane with the origin removed. That is: [Chapter 8.5 Review]

$$A = \{(x,y) \mid x,y \in \mathbb{R}\} - \{(0,0)\}\,.$$

Define a relation on A by the rule:

$$(a,b)R(c,d) \leftrightarrow (a,b) \text{ and } (c,d) \text{ lie on the same line through the origin}\,.$$

a. Prove that R is an equivalence relation.

Answer: R is reflexive because (a, b) and (a, b) lie on the same line through the origin, namely on the line $y = b/x$ (if $a \neq 0$), or else on the line $x = 0$ (if $a = 0$).

R is symmetric. Suppose (a, b) and (c, d) lie on the same line through the origin, then (c, d) and (a, b) lie on the same line through the origin.

R is transitive. Suppose (a, b) and (c, d) lie on the same line L through the origin and (c, d), (e, f) lie on the same line M through the origin. The L and M both contain the two distinct points $(0, 0)$ and (c, d). Therefore L and M are the same line, and this line contains (a, b) and (e, f). Therefore (a, b) and (e, f) line on the same line through the origin.

Note: The proof that R is an equivalence relation can be carried out using analytic geometry: if (a, b) and (c, d) lie on the same nonvertical line through the origin, then the slope must equal b/a because the line passes through $(0, 0)$ and (a, b) and the slope must also equal d/c because the line passes through $(0, 0)$ and (c, d); thus, $b/a = d/c$, or $ad = bc$. If (a, b) and (c, d) lie on the same vertical line through the origin, then the points must have the form $(0, b)$ and $(0, d)$, and again it must happen that $ad = bc$. Therefore, $(a, b)R(c, d)$ means that $ad = bc$. This equation can be used to verify that R is reflexive, symmetric, and transitive.

b. Describe the equivalence classes arising from the equivalence relation R.

Answer: Each equivalence class is the set of points of A on a line of the form $y = mx$ or the vertical line $x = 0$, with the origin removed.

c. If A is replaced by the entire plane, is R an equivalence relation?

Answer: If A is replaced by the entire plane, R is not an equivalence relation. It fails to satisfy the transitive property; for example, $(1, 2)R(0, 0)$ and $(0, 0)R(2, 2)$, but $(1, 2)R/(2, 2)$ because the line passing through $(1, 2)$ and $(2, 2)$ does not pass through the origin.

Some more problems on the 8.1, and 8.3

Question: Deal with these relations on the set of real numbers: [Chapter 8.1 Review]

$R_1 = \{(a, b) \in R^2 \mid a > b\}$, the "greater than" relation,

$R_2 = \{(a, b) \in R^2 \mid a \geq b\}$, the "greater than or equal to" relation,

$R_3 = \{(a, b) \in R^2 \mid a < b\}$, the "less than" relation,

$R_4 = \{(a, b) \in R^2 \mid a \geq b\}$, the "less than or equal to" relation,

$R_5 = \{(a, b) \in R^2 \mid a = b\}$, the "equal to" relation,

$R_6 = \{(a, b) \in R^2 \mid a \neq b\}$, the "unequal to" relation,

b. $R_1 \cup R_5$

Answer: The union of two relations is the union of these sets. Thus $R_1 \cup R_5$ holds between two real numbers if R_1 holds or R_5 holds (or both, it goes without saying). Here this means that the first number if greater than or equals the second –This is just relation R_2.

d. $R_3 \cup R_5$

Answer: The intersection of two relations is the intersection of these sets. Thus $R_3 \cap R_5$ holds between two relations if R_3 holds and R_5 holds as well. Thus for (a, b) to be in $R_3 \cap R_5$, we must have $a < b$ and $a = b$. So the answer is $\emptyset$.

f. $R_2 - R_1$

Answer: Recall that $R_2 - R_1 = R_2 \cap \overline{R_1}$. But $\overline{R_1} = R_4$, so we are asked for $R_2 \cap R_4$. It is to say $a \geq b$ and $a \leq b$. This happens precisely when $a = b$, which is relations R_5

g. $R_1 \oplus R_3$

Answer: Recall that $R_1 \oplus R_3 = (R_1 \cap \overline{R_3}) \cup (R_3 \cap \overline{R_1})$. We see that $R_1 \cap \overline{R_3} = R_1 \cap R_2 = R_1$ and $R_3 \cap \overline{R_1} = R_3 \cap R_4 = R_3$. Thus our answer if $R_1 \cup R_3 = R_6$.

h. $R_2 \oplus R_4$

Answer: Recall that $R_2 \oplus R_4 = (R_2 \cap \overline{R_4}) \cup (R_4 \cap \overline{R_2})$. We see that $R_2 \cap \overline{R_4} = R_2 \cap R_1 = R_1$ and $R_4 \cap \overline{R_2} = R_4 \cap R_3 = R_3$. Thus our answer if $R_1 \cup R_3 = R_6$.

Question: Determine whether the relations represented by the matrices as following are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive. [Chapter 8.3 Review]

Review For reflexivity we want all 1's on the main diagonal; for irreflexivity we want all 0's on the main diagonal; for symmetry, we want the matrix to be symmetric about the main diagonal (equivalently, the matrix equals its transpose); for antisymmetry we want there never to be two 1's symmetrically placed about the main diagonal (equivalently, the meet of the matrix and its transpose has no 1's off the main diagonal); and for transitivity, we want the Boolean square of the matrix (the Boolean product of the matrix and itself) to be "less than or equal to" the original matrix in the sense that there is a 1 in the original matrix at every location where there is a 1 in the Boolean square.

a. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Answer: Since some 1's and some 0's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is antisymmetric – look at positions $(1, 2)$ vs $(2, 1)$, $(1, 3)$ vs $(3, 1)$. Finally the relation is not transitive; for example, the 1's in positions $(2, 1)$ and $(1, 3)$ would require a 1 in position $(2, 3)$ if the relation were to be transitive.

b. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: Since there are all 1's on the main diagonal, this relation is reflexive and not irreflexive. The relation is symmetric and not antisymmetric since 1's and 0's are symmetrically placed with respect to the main diagonal. Finally, the Boolean square of this matrix is itself, so the relation is transitive.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$