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ICS 241

Recitation Lecture Note #8

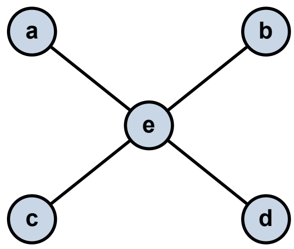
October 16, 2009

Recitation #8

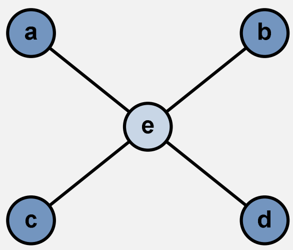
Question: Determine whether the graph is bipartite. [Chapter 9.2 Review]

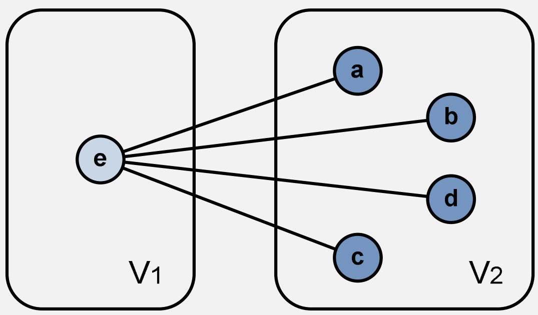
Review: A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

a.

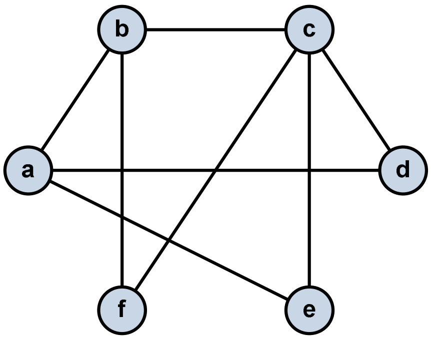


Answer: If we color vertex a in dark blue, vertex e needs to stay in light blue because they are adjacent. Since vertices b, d, and c are all adjacent to e, they can't stay in light blue, and because we have only two colors to choose from, vertices b, d and c need to be colored in dark blue. Now we look at the output graph to see if there's any adjacent vertices has the same color. The answer is no, so this graph is a bipartite.

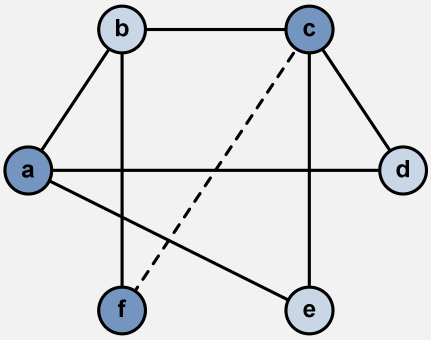




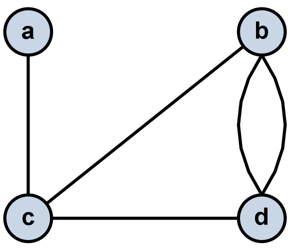
b.



Answer: If we color vertex a in dark blue, vertex $b, d,$ and e needs to stay in light blue because they are adjacent to a. Since vertices c and f are both adjacent to b, they can't stay in light blue, and because we have only two colors to choose from, vertices c and f need to be colored in dark blue. Now we look at the output graph to see if there's any adjacent vertices has the same color. The answer is yes. Vertices c and f are adjacent to each other and have the same color, so this graph is <u>*not*</u> a bipartite. As we can see edge cf violates the rule that there should be no edge in G connecting two vertices in the same disjoint set of a bipartite.



Question: Determine the adjacency matrix and the incidence matrix for the following graph. [Chapter 9.3 Review]

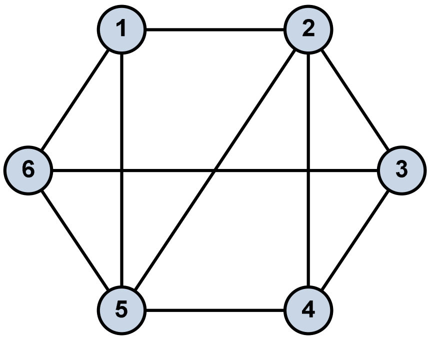


Answer:

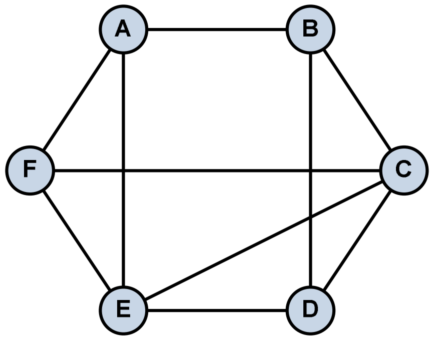
$$M_{adjacent} = \begin{array}{c|c} & \begin{array}{cccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array} \qquad M_{incidence} = \begin{array}{c|c} & \begin{array}{ccccc} \underline{ac} & \underline{bc} & \underline{cd} & \underline{db} & \underline{bd} \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Question: Determine whether the following graphs are isomorphic. [Chapter 9.3 Review]

A.



B.



Answer: We go through the following checklist that might tell us immediately if the two are not isomorphic:

Do they have the same number of vertices? Yes, 6.

Do they have the same number of edges? Yes, 9.

Do they have matching degree sequences? Yes, $(3, 4, 3, 3, 4, 3)$, $(3, 3, 4, 3, 4, 3)$.

Do they have the same number of 3-cycles? Yes, both contain four 3-cycles.

This list can keep going. Since there is no obvious reason to think they are not isomorphic, we try to construct an isomorphism, f (I'll use a connecting line to show this mapping). Note that the above does not tell us there is an isomorphism, only that there might be one.

In the graph on the left, only vertices 2 and 5 have degree four. In the graph on the right, only vertices C and E have degree four. Therefore, if the two graphs are to be isomorphic, , we must have 2 and 5 correspond to C and E as either $2 - C, 5 - E$, or as $2 - E, 5 - C$. In this example, either correspondence gives rise to an isomorphism:

$$1 - F, 2 - C, 3 - B, 4 - D, 5 - E, 6 - A.$$

$$1 - D, 2 - E, 3 - A, 4 - F, 5 - C, 6 - B.$$

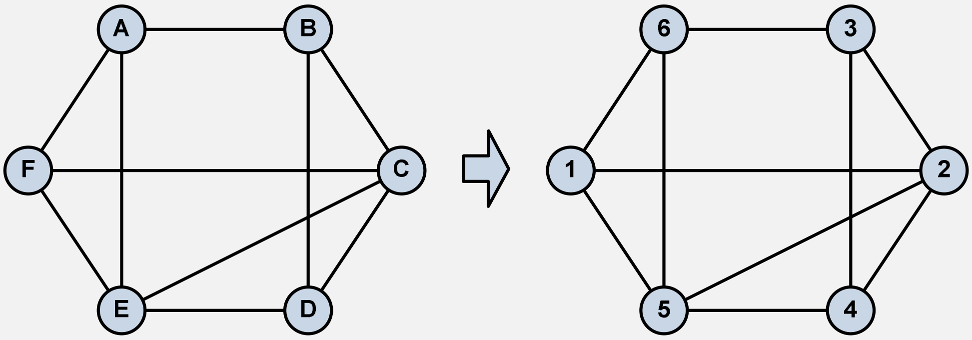
It is possible only one choice would work, or both choices may work, or neither choice may work, which would tell us there is no isomorphism. Let try the first one: $1 - F, 2 - C, 3 - B, 4 - D, 5 - E, 6 - A$:

We have adjacency matrix for A, where we list the vertices in the ascending order:

$$M_{adjacency_A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Since the mapping from graph A to graph B, we think is $1-F, 2-C, 3-B, 4-D, 5-E, 6-A$, we reconstruct the appearance of the graph B:

B. B'.



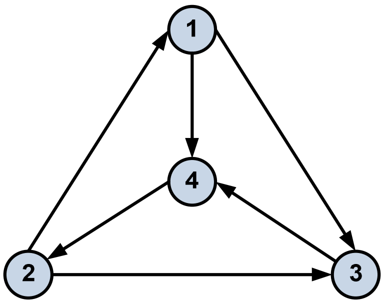
$$\Rightarrow M_{adjacency_{B'}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} = M_{adjacency_A}$$

Since $M_{adjacency_{B'}} = M_{adjacency_A}$, and adjacency is preserved graph invariant, graphs A and graph B are isomorphic.

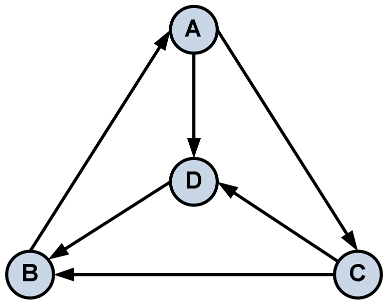
Note: The advantage of the checklist is that it will give you a quick and easy way to show two graphs are not isomorphic if some invariant of the graphs turn out to be different. If you examine the logic, however, you will see that if two graphs have all of the same invariants we have listed so far, we still wouldn't have a proof that they are isomorphic. Indeed, there is no known list of invariants that can be efficiently checked to determine when two graphs are isomorphic. The best algorithms known to date for determining graph isomorphism have exponential complexity (in the number n of vertices).

Question: Determine whether the following graphs are isomorphism. [Chapter 9.3 Review]

$A.$



$B.$



Answer: We go through the following checklist that might tell us immediately if the two are not isomorphic:

Do they have the same number of vertices? Yes, 4.

Do they have the same number of edges? Yes, 6.

Do they have matching in-degree (deg^{-}) sequences? Yes, $(1, 1, 2, 2)$, $(1,2, 1, 2)$.

Do they have matching out-degree (deg^{+}) sequences? Yes, $(2, 2, 1, 1)$, $(2, 1, 2, 1)$.

Do they have the same number of 3-cycles? Yes, both contain two 3-cycles.

This list can keep going. Since there is no obvious reason to think they are not isomorphic, we try to construct an isomorphism, f (I'll use a connecting line to show this mapping). Note that the above does not tell us there is an isomorphism, only that there might be one.

In the graph on the left, only vertices 3 and 4 have in-degree two. In the graph on the right, only vertices B and D have in-degree two. Therefore, if the two graphs are to be isomorphic, , we must have 3 and 4 correspond to B and D as either $3 - B, 4 - D$, or as $3 - D, 4 - B$. Therefore we come up with the following isomorphism(keep in mind they are just guesses at this point) :

$$1 - C, 2 - A, 3 - B, 4 - D.$$

$$1 - C, 2 - A, 3 - D, 4 - B.$$

$$1 - A, 2 - C, 3 - B, 4 - D.$$

$$1 - A, 2 - C, 3 - D, 4 - B.$$

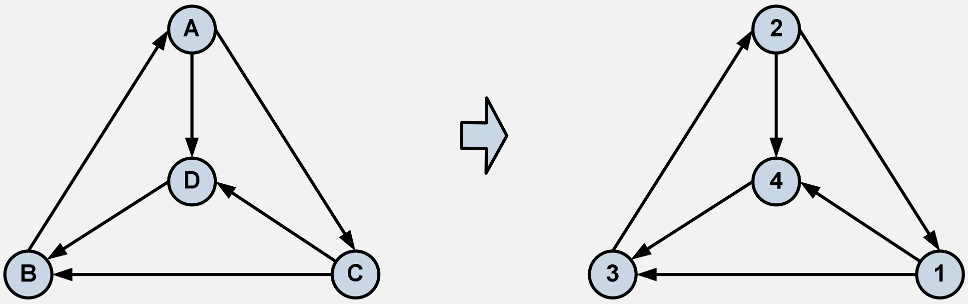
It is possible only one choice would work, or both choices may work, or neither choice may work, which would tell us there is no isomorphism. Let try the first one: $1 - C, 2 - A, 3 - B, 4 - D$.

We have adjacency matrix for A, where we list the vertices in the ascending order:

$$M_{adjacency_A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Since the mapping from graph A to graph B, we think is $1-C, 2-A, 3-B, 4-D$. we reconstruct the appearance of the graph B:

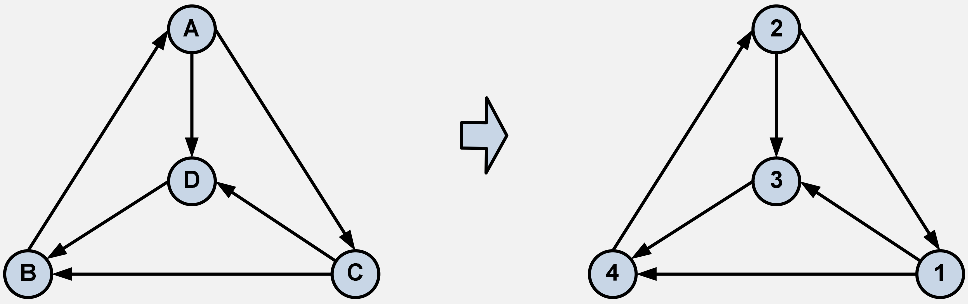
B. B'.



$$\Rightarrow M_{adjacency_{B'}} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \neq M_{adjacency_A}$$

So $1-C, 2-A, 3-B, 4-D$ was not a solution. Let's move on to our second guess, $1-C, 2-A, 3-D, 4-B$. We reconstruct the appearance of the graph B:

B. B''.

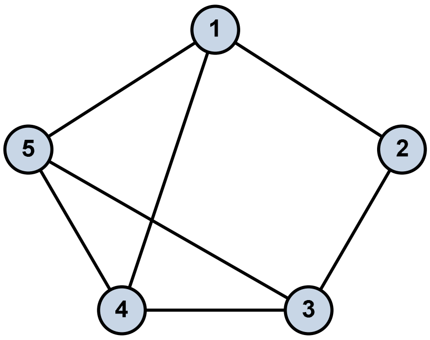


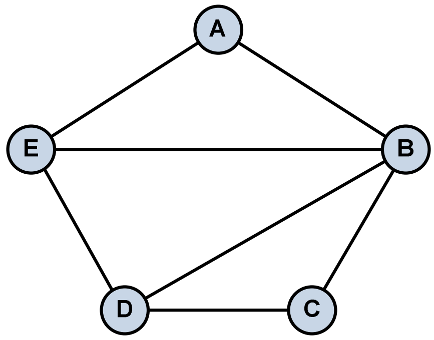
$$\Rightarrow M_{adjacency_{B''}} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = M_{adjacency_A}$$

Since $M_{adjacency_{B''}} = M_{adjacency_A}$, and adjacency is preserved graph invariant, graphs A and graph B are isomorphic.

Question: Determine whether the following graphs are isomorphism. [Chapter 9.3 Review]

A. B.





Answer: We go through the following checklist that might tell us immediately if the two are not isomorphic:

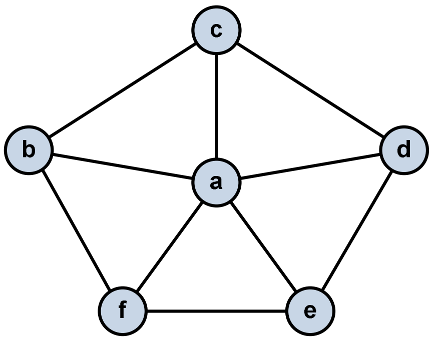
Do they have the same number of vertices? Yes, 5.

Do they have the same number of edges? Yes, 7.

Do they have matching in-degree sequences? No, $(3, 2, 3, 3, 3)$, $(2, 4, 2, 3, 3)$.

So by going through the graph invariant check list, we found one graph invariant appear differently in graph A and graph B. Therefore these two graphs are not isomorphism.

Question: Use powers of the adjacency matrix to find the following numbers of paths in the given graph below. [Chapter 9.4 Review]



a. path from b to e of length 3

Answer: Using alphabetical order to determine the position of the rows and columns, we have the adjacency matrix

$$M_{adjacency} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow M_{adjacency}^3 = \begin{bmatrix} 10 & 9 & 9 & 9 & 9 & 9 \\ 9 & 4 & 7 & 5 & 5 & 7 \\ 9 & 7 & 4 & 7 & 5 & 5 \\ 9 & 5 & 7 & 4 & 7 & 5 \\ 9 & 5 & 5 & 7 & 4 & 7 \\ 9 & 7 & 5 & 5 & 7 & 4 \end{bmatrix}.$$

The 2-5 entry represent the paths from b to e. The number of paths from vertex b to vertex e with a length of 3 is, therefore, 5.

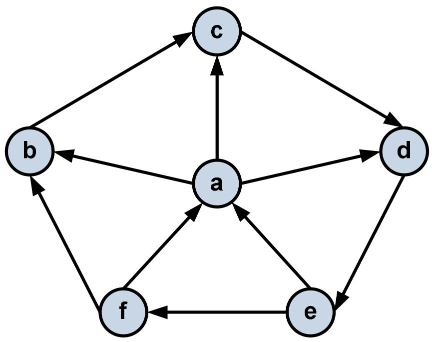
b. path from a to c of length 5

Answer: Using alphabetical order to determine the position of the rows and columns, we have the adjacency matrix

$$M_{adjacency} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow M_{adjacency}^5 = \begin{bmatrix} 140 & 101 & 101 & 101 & 101 & 101 \\ 101 & 64 & 72 & 67 & 67 & 72 \\ 101 & 72 & 64 & 72 & 67 & 67 \\ 101 & 67 & 72 & 64 & 72 & 67 \\ 101 & 67 & 67 & 72 & 64 & 72 \\ 101 & 72 & 67 & 67 & 72 & 64 \end{bmatrix}.$$

The 1-3 entry represent the paths from a to c. The number of paths from vertex a to vertex c with a length of 5 is, therefore, 101.

Question: Use powers of the adjacency matrix to find the following numbers of paths in the given graph below. [Chapter 9.4 Review]



a. path from e to c of length 3

Answer: Using alphabetical order to determine the position of the rows and columns, we have the adjacency matrix

$$M_{adjacency} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow M_{adjacency}^3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

The 5-3 entry represent the paths from e to c. The number of paths from vertex e to vertex c with a length of 3 is, therefore, 3.

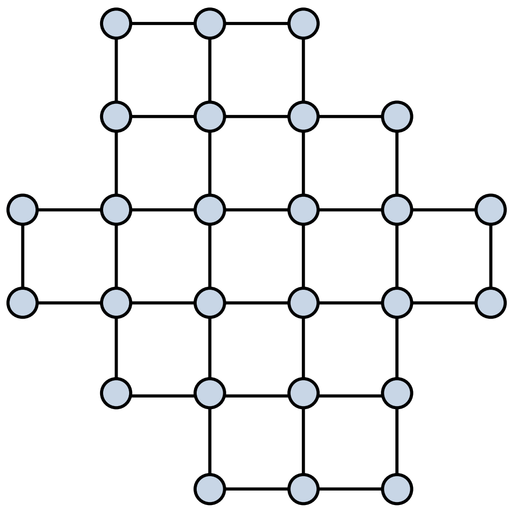
b. path from e to e of length 4

Answer: Using alphabetical order to determine the position of the rows and columns, we have the adjacency matrix

$$M_{adjacency} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow M_{adjacency}^4 = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 1 & 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

The 5-5 entry represent the paths from e to e. The number of paths from vertex e to vertex e with a length of 4 is, therefore, 2.

Question: Given the following graph, answer the questions below. [Chapter 9.5 Review]



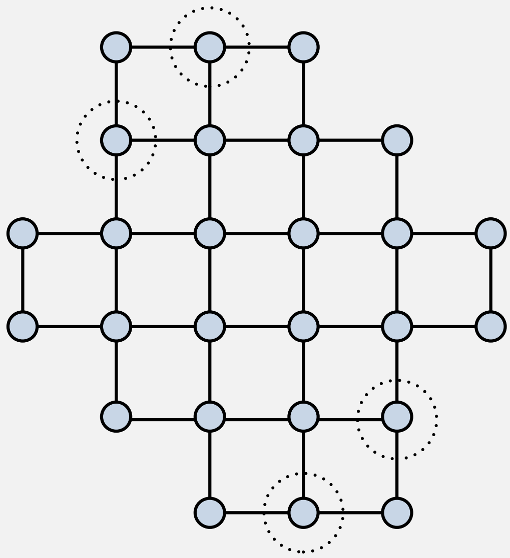
Review: A Euler circuit in graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

Theorem 1. A connected multigraph with at least two vertices has and Euler circuit if and only if each of its vertices has even degree.

Theorem 2. A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

a. Determine whether the following graph has an Euler circuit or Euler path.

Answer: The graph has no Euler circuit because there are vertices with degrees of 3. The graph has no Euler path either because it has four vertices of odd degree.



b. Determine whether the following graph has a Hamilton circuit or Hamilton path

Review: A Hamilton circuit is a circuit that traverses each vertex of a graph G exactly once. A Hamilton path is a path that traverses each vertex in a graph G exactly once.

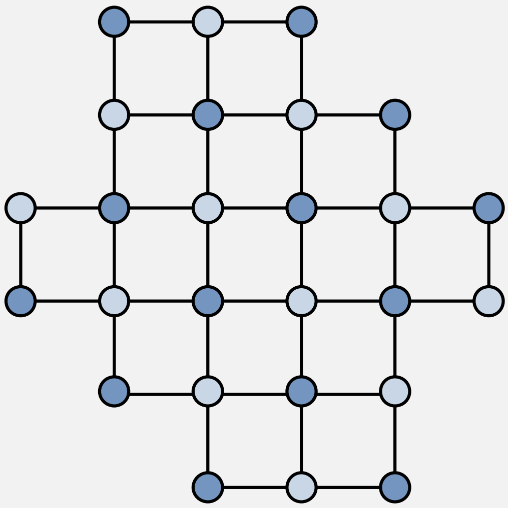
Dirac's Theorem. If (but not only if) G is connected, simple, has $n \geq 3$ vertices and $\forall v, \deg(v) \geq n/2$, then G has a Hamilton circuit.

Ore's Theorem. If (but not only if) G is connected, simple, has $n \geq 3$ vertices, and $\deg(u) + \deg(v) \geq n$ for every pair u and v of non-adjacent vertices, then G has a Hamilton circuit.

Answer: The graph does not satisfy either of the two rules above. But since the rules do not provide necessary conditions for the existence of a Hamilton circuit, we still don't have a conclusion. Though you might already have an answer by moving your finger over the graph, we are going to reason it logically as the following.

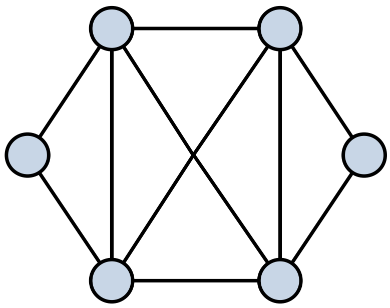
The graph has no Hamilton circuit or Hamilton path. To see this, note that the graph is bipartite. If the vertices are colored dark blue (starting with the left vertex at the top of the figure) and light blue so that no adjacent vertices have the same color, there are 14 vertices in dark blue and 12 in light blue nodes. Any Hamilton circuit or path must consist of an alternating sequence of dark blue and light blue vertices, which is not possible with two more dark blue nodes than the light blue nodes.

In other words, in a bipartite, in order to have a Hamilton circuit of path, the numbers of vertices in the two disjoint sets need to be either the same or off by one. Satisfying this, however, does not mean it for sure has a Hamilton circuit or path though.



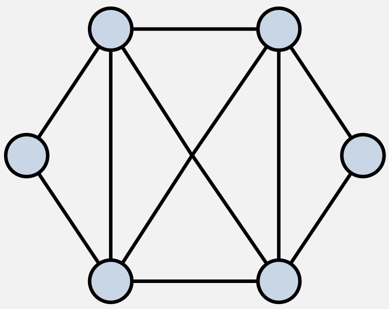
Question: Find a Euler circuit and path in the following graphs. [Chapter 9.5 Review]

a. Look for an Euler circuit

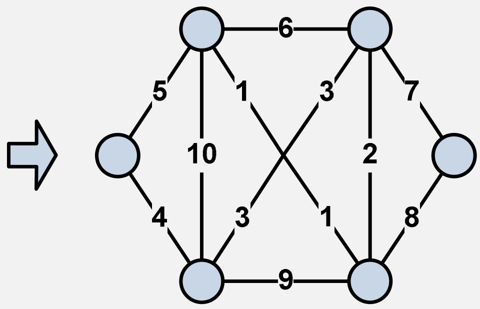


Answer:

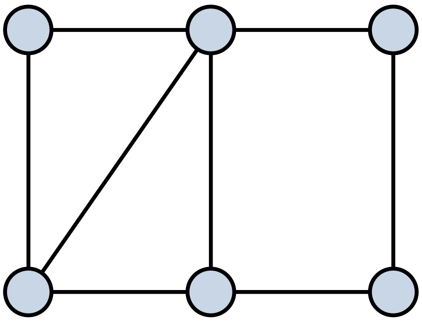
Input graph



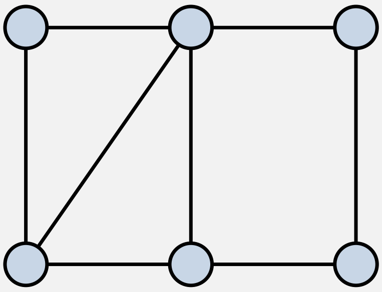
Euler circuit



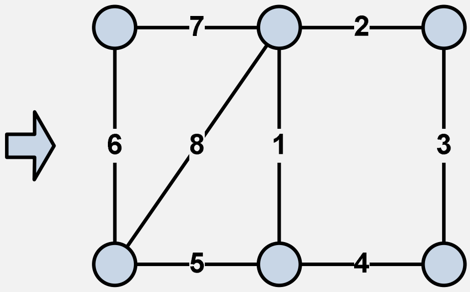
b. Look for an Euler path



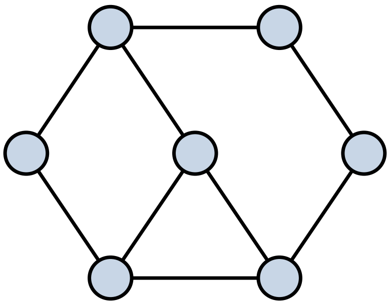
Answer: Input graph



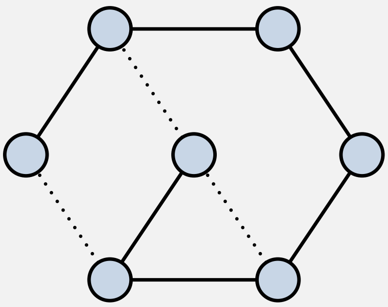
Euler path



Question: Find a Hamilton circuit and path in the following graph. [Chapter 9.5 Review]



Answer: Hamilton path



Hamilton cricuit