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ICS 241

Recitation Lecture Notes #13

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Question: Find a Boolean product of the Boolean variable x, y, and z, or their complements, that has the value 1 if and only the following condition holds. [Chapter 11.2 Review]

a. $x = y = 0, z = 1$

Answer: $\bar{x}\bar{y}z$.

b. $x = 0, y = 1, z = 0$

Answer: $\bar{x}y\bar{z}$.

c. $x = 0, y = z = 1$

Answer: $\bar{x}yz$.

Review: $\bar{\bar{x}} = x$ Law of the double complement

$x + x = x$ and $x \cdot x = x$	Idempotent laws
$x + 0 = x$ and $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ and $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ and $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ and $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \bar{x} + \bar{y}$ and $\overline{(x + y)} = \bar{x}\bar{y}$	De Morgan's laws
$x + xy = x$ and $x(x + y) = x$	Absorption laws
$x + \bar{x} = 1$	Unit property
$x\bar{x} = 0$	Zero property

Question: Find the sum-of-products expansions of these Boolean functions. [Chapter 11.2 Review]

a. $F(x, y) = x + y$

Answer: We can rewrite the function as $F(x, y) = x \cdot 1 + y \cdot 1$. One way of doing this is,

$$
\begin{aligned}
\mathrm{F(x,y)} &= \mathrm{x} \cdot 1 + \mathrm{y} \cdot 1 \\
&= \mathrm{x(y + \bar{y}) + y(x + \bar{x})} && \text{Unit property} \\
&= \mathrm{xy + x\bar{y} + yx + y\bar{x}} \\
&= xy + \mathrm{x\bar{y}} + xy + \mathrm{y\bar{x}} && \text{Commutative law} \\
&= xy + \mathrm{x\bar{y}} + \mathrm{y\bar{x}}\,. && \text{Idempotent law}
\end{aligned}
$$

Note at the second step when $x(y + \bar{y}) + y(x + \bar{x})$ is on the right hand side of the equation, the expansion of the expression can be huge. Then we have a short cut to derive the conclusion rather than calculating one step at a time.

On fact we know is that we have only two Boolean variables. The expressions of the "products" are very limited. In this case we have only four. They are xy, $x\bar{y}$, $\bar{x}y$, and $\overline{xy}$. Based on the Idempotent law, the occurrence of each expression doesn't affect. Then, for the sum-of-products, all we need to do, is going over the product expression list, and decide whether each of the expression are in the summation.

For this example, we look at $x(y + \bar{y}) + y(x + \bar{x})$. We can quickly decide, that xy, $x\bar{y}$, and $y\bar{x}$ are in the summation, while $\overline{xy}$ should not be presented. Therefore, we have the conclusion,

$$
\begin{aligned}
F(x, y) &= x + y \\
&= xy + x\bar{y} + y\bar{x}\,.
\end{aligned}
$$

A third way of doing it is of course using the element table,

x	y	$F(x, y)$
1	1	1
1	0	1
0	1	1
0	0	0

The sum-of products expansion for $F(x, y) = x + y$ is shown in the first three rows,

$$
\begin{aligned}
F(x, y) &= x + y \\
&= xy + x\bar{y} + y\bar{x}\,.
\end{aligned}
$$

b. $F(x, y, z) = x + y + z$

Answer: We can rewrite the function as below and derive the sum-of-products using the second method presented above.

$$F(x, y, z) = x \cdot 1 \cdot 1 + y \cdot 1 \cdot 1 + z \cdot 1 \cdot 1$$
$$= x(y + \bar{y})(z + \bar{z}) + y(x + \bar{x})(z + \bar{z}) + z(x + \bar{x})(y + \bar{y})$$

Now, instead of expending the expression above, we list out all the possible Boolean products for these three Boolean variables. They are,

$$xyz, \quad \bar{x}yz, \quad x\bar{y}z, \quad xy\bar{z}, \quad \bar{x}\bar{y}z, \quad x\bar{y}\bar{z}, \quad \bar{x}y\bar{z}, \quad \bar{x}\bar{y}\bar{z}$$

Evaluate the equation's right hand side. It's not hard to decide which of the Boolean products should be presented in the sum-of-products expression.

$$F(x, y, z) = x + y + z$$
$$= x(y + \bar{y})(z + \bar{z}) + y(x + \bar{x})(z + \bar{z}) + z(x + \bar{x})(y + \bar{y})$$
$$= xyz + \bar{x}yz + x\bar{y}z + xy\bar{z} + \bar{x}\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z}\,.$$

Also, keep in mind that based on the Commutative law, $x + y = y + x$ and $xy = yx$. This ensures our computation above.

c. $F(x, y, z) = (x + z)y$

Answer: We can rewrite the function as below and derive the sum-of-products using the second method presented above.

$$F(x, y, z) = (x + z)y$$
$$= xy \cdot 1 + zy \cdot 1$$
$$= xy \cdot (z + \bar{z}) + zy \cdot (x + \bar{x})$$

Now, instead of expending the expression above, we list out all the possible Boolean products for these three Boolean variables. They are, (of course the same as above)

$$xyz, \quad \bar{x}yz, \quad x\bar{y}z, \quad xy\bar{z}, \quad \bar{x}\bar{y}z, \quad x\bar{y}\bar{z}, \quad \bar{x}y\bar{z}, \quad \bar{x}\bar{y}\bar{z}$$

Evaluate the equation's right hand side. It's not hard to decide which of the Boolean products should be presented in the sum-of-products expression.

$$F(x, y, z) = (x + z)y = xy \cdot (z + \bar{z}) + zy \cdot (x + \bar{x})$$
$$= xyz + \bar{x}yz + xy\bar{z}\,.$$

Also, keep in mind that based on the Commutative law, $x + y = y + x$ and $xy = yx$. This ensures our computation above.

d. $F(x, y, z) = x$

Answer: We can rewrite the function as below and derive the sum-of-products using the second method presented above.

$$\begin{aligned} F(x, y, z) &= x \\ &= x \cdot 1 \cdot 1 \\ &= x(y + \bar{y})(z + \bar{z}) \end{aligned}$$

Now, instead of expending the expression above, we list out all the possible Boolean products for these three Boolean variables. They are, (of course the same as above)

$$xyz, \quad \bar{x}yz, \quad x\bar{y}z, \quad xy\bar{z}, \quad \bar{x}\bar{y}z, \quad x\bar{y}\bar{z}, \quad \bar{x}y\bar{z}, \quad \bar{x}\bar{y}\bar{z}$$

Evaluate the equation's right hand side. It's not hard to decide which of the Boolean products should be presented in the sum-of-products expression.

$$\begin{aligned} F(x, y, z) &= x \\ &= x(y + \bar{y})(z + \bar{z}) \\ &= xyz + x\bar{y}z + xy\bar{z} + x\bar{y}\bar{z}\,. \end{aligned}$$

e. $F(x, y, z) = x\bar{y}$

Answer: We can rewrite the function as below and derive the sum-of-products using the second method presented above.

$$\begin{aligned} F(x, y, z) &= x\bar{y} \\ &= x\bar{y} \cdot 1 \\ &= x\bar{y}(z + \bar{z}) \end{aligned}$$

Now, we want to decide which method is easier, and progress with the easier approach. Clearly, just expand the expression is easier in this example. So, we continue.

$$\begin{aligned} F(x, y, z) &= x\bar{y} \\ &= x\bar{y}(z + \bar{z}) \\ &= x\bar{y}z + x\bar{y}\bar{z}\,. \end{aligned}$$

Question: Find a Boolean function $F(x, y, z)$ that has the following element table. [Chapter 11.2 Review]

x	y	z	$F(x,y,z)$
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

Answer: Consider the second row of the table of values for f, where $x = 1, y = 1, z = 0$. In this case the product $xyz = 1 \cdot 1 \cdot 0 = 1 \cdot 1 \cdot 1 = 1$. Next, consider the third row of the table, where $x = 1, y = 0, z = 1$. In this case the product $xyz = 1 \cdot 1 \cdot 1 = 1$. In general, we can obtain a function value 1 in a particular row if we form the appropriate product of literals. We can then add these products to obtain any function table. In this case, we form the sum as the formula for F,

$$F(x, y, z) = xy\bar{z} + x\bar{y}z + \bar{x}y\bar{z} + \bar{x}\bar{y}z\,.$$

Question: Let $F(x, y, z) = \bar{z} + \bar{x}z$. [Chapter 11.2 Review]

a. Find the sum-of-products expansion, Disjunctive Normal Form (DNF).

Answer: We first derive the following element table,

x	y	z	$F(x,y,z)$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

The sum-of-products expansion for F is,

$$F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}\,.$$

b. Find the product-of-sums expansion, Conjunctive Normal Form (CNF).

Answer: Evaluate the element table above, we have

$$F(x,y,z) = (\bar{x} + \bar{y} + \bar{z})(x + \bar{y} + z)\,.$$

Question: Find the sum-of-products expansion of the Boolean function $F(w,x,y,z)$ that equals 1 if and only if an odd number of $w, x, y,$ and z have the value 1. [Chapter 11.2 Review]

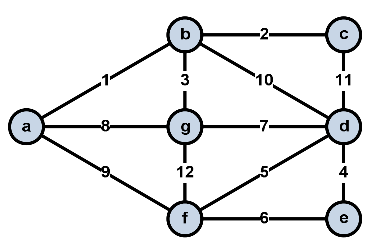
Answer:

w	x	y	z	F
1	1	1	1	0
1	1	1	0	1
1	1	0	1	1
1	1	0	0	0
1	0	1	1	1
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1
0	1	1	1	1
0	1	1	0	0
0	1	0	1	0
0	1	0	0	1
0	0	1	1	0
0	0	1	0	1
0	0	0	1	1
0	0	0	0	0

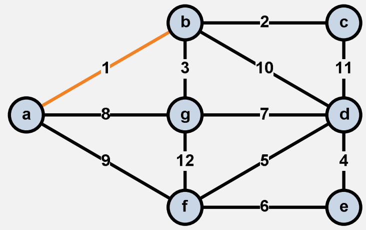
What we are seeing in this element table is that the highlighted rows satisfy the Boolean function $F = 1$. Therefore, we have the corresponding Boolean products expansion of F,

$$F(w,x,y,z) = wxy\bar{z} + wx\bar{y}z + w\bar{x}yz + w\bar{x}\bar{y}\bar{z} + \bar{w}xyz + \bar{w}x\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}z\,.$$

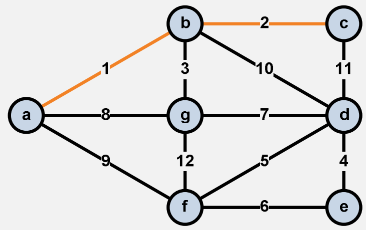
Question: Use Prim's algorithm to find a minimum spanning tree for the given weighted graph. [Chapter 10.5 Review]



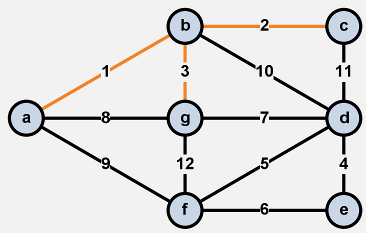
Answer: Step 1. Start from the smallest weight edge $\{a, b\}$



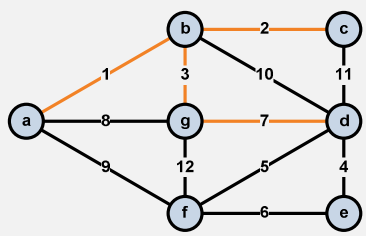
Step 2. Append $\{b, c\}$



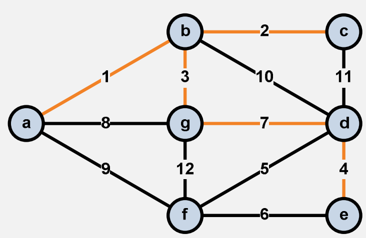
Step 3. Append $\{b, g\}$



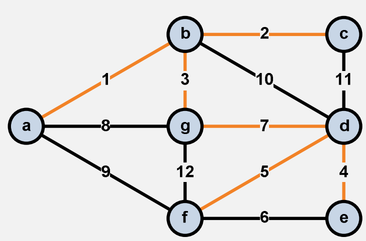
Step 4. Append $\{g, d\}$. Note that $\{d, e\} = 4$ is less than $\{g, d\} = 7$, but $\{d, e\}$ is not adjacent.



Step 5. Append $\{d, e\}$

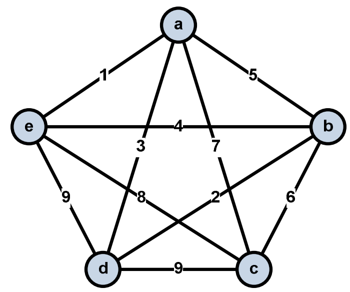


Step 6. Append $\{d, f\}$

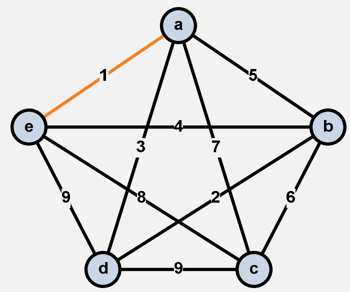


At this point all vertices are traversed. Prim's algorithm terminates execution, and results a minimum spanning tree with a total weight of 22.

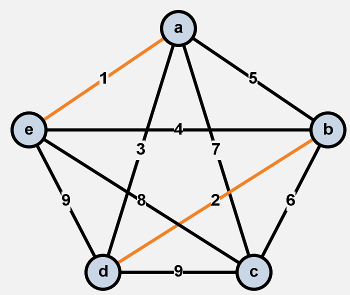
Question: Use Kruskal's algorithm to find a minimum spanning tree for the given weighted graph. [Chapter 10.5 Review]



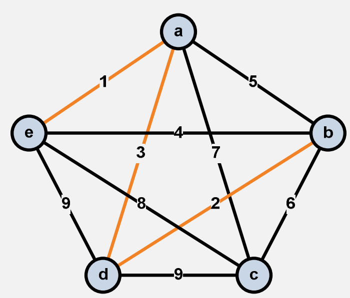
Answer: Step 1. Start from the smallest weight edge $\{a, e\}$



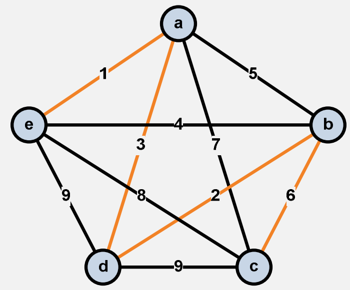
Step 2. Append $\{b, d\}$



Step 3. Append $\{a, d\}$



Step 4. Append $\{b, c\}$

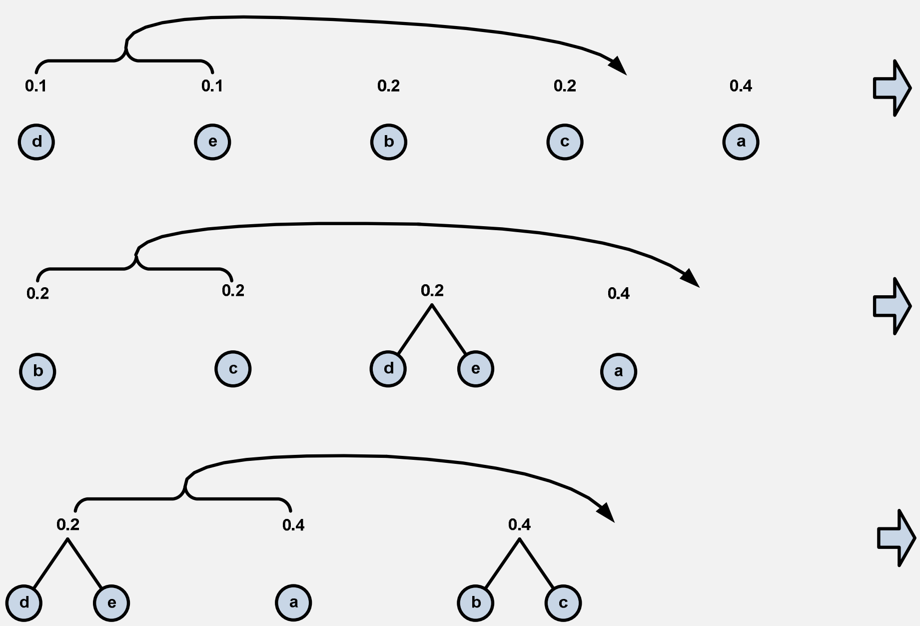


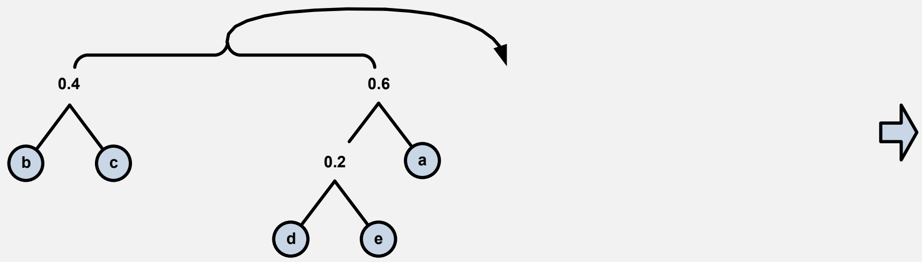
At this point all vertices are traversed. Kruskal's algorithm terminates execution, and results a minimum spanning tree with a total weight of 12.

Question: Huffman encoding problem [Chapter 10.2 Review]

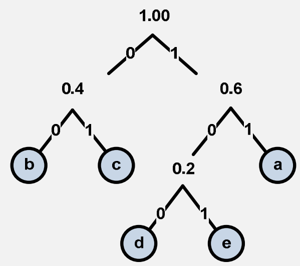
a. Use Huffman coding to encode these symbols with frequencies a: 0.4, b: 0.2, c: 0.2, d: 0.1, e: 0.1 in two different ways by breaking ties in the algorithm differently. First, among the trees of minimum weight select two trees with the smallest number of vertices at each stage. Second, among the trees of minimum weight select two trees with the largest number of vertices to combine at each stage of the algorithm.

Answer: In the first way of constructing the Huffman tree, we break ties by placing the new node at the last position.

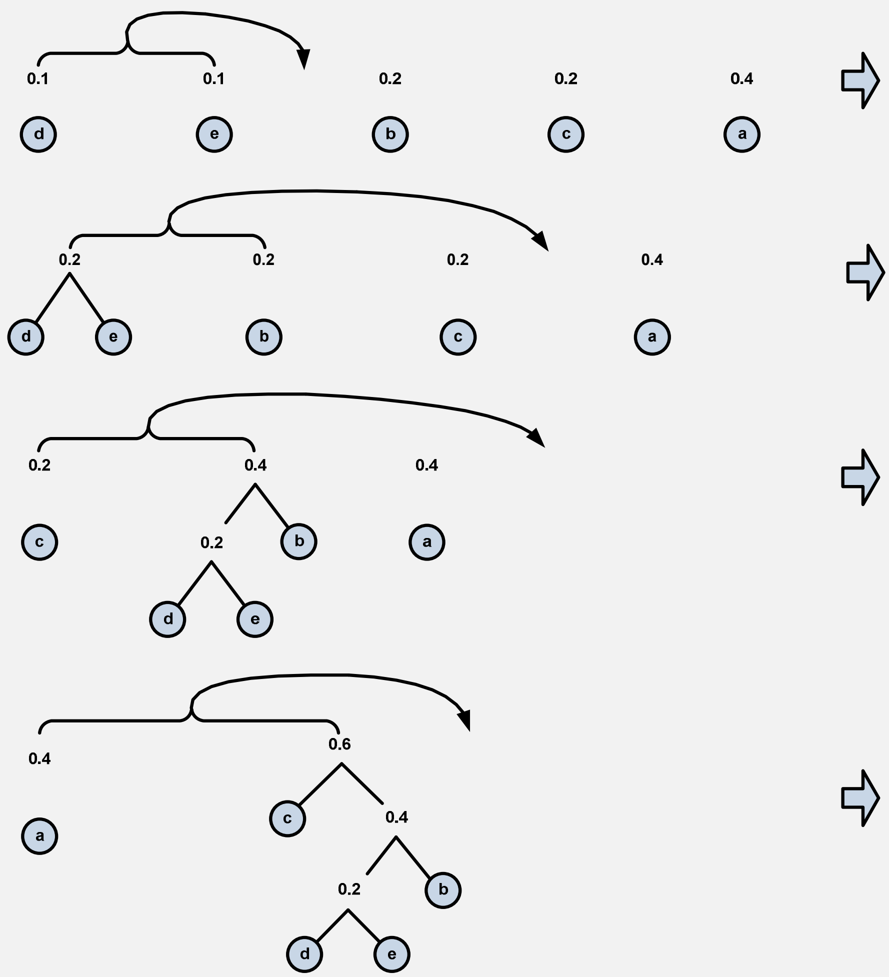




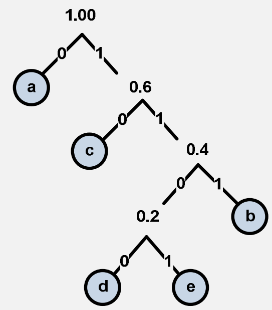
We obtain the following Huffman tree, and therefore, obtain the following Huffman codes for the 5 symbols: a: 11, b: 00, c: 01, d: 100, e: 101.



In the second way of constructing the Huffman tree, we break ties by placing the new node at the first position.



We obtain the following Huffman tree, and therefore, obtain the following Huffman codes for the 5 symbols: a: 0, b: 111, c: 10, d: 1100, e: 1101.



b. Compute the average number of bits required to encode a symbol with each code.

Answer: The average of the first method is $2 \cdot 0.4 + 2 \cdot 0.2 + 2 \cdot 0.2 + 3 \cdot 0.1 + 3 \cdot 0.1 = 2.2$, and the average for the second method is: $1 \cdot 0.4 + 3 \cdot 0.2 + 2 \cdot 0.2 + 4 \cdot 0.1 + 4 \cdot 0.1 = 2.2$. We knew ahead of time, of course, that these would turn out to be equal, since the Huffman algorithm minimizes the expected number of bits.